

AN ANALYSIS OF THE SEIDEL LAPLACIAN ENERGY OF A FUZZY INTUITIONISTIC SYSTEM

Sivaranjani Krishnaraj^{1,2,*}, and Shanmuga Sundaram Olappalyam Vaiyapuri²

Received: October 25, 2024; Revised: December 01, 2024; Accepted: February 17, 2025

Abstract

This paper presents one of the latest research works in intuitionistic fuzzy graph theory. Along with questionable concepts conveyed in distinctive languages, intuitionistic fuzzy set theory offers a noteworthy and ground-breaking depiction of vulnerability estimation. The concept of energy is related to the spectrum of a graph. The energy of graphs plays a vital role in graph theory. In mathematics, the total sum of the absolute values of the eigenvalues of the graph's adjacency matrix is referred to as the graph's energy. In the framework of spectral graph theory, this quantity is extensively studied. When analyzing various repulsive and appealing circumstances, energy graphs are a great tool. Analogous energies based on the eigenvalues of various additional graph matrices have been examined recently. Graph energy possesses numerous mathematical characteristics which are investigated in this paper. One type of graph energy is Seidel Laplacian energy. The Seidel Laplacian energy (SLE) of a graph has already been extended to fuzzy graphs. In this manuscript, it is extended to an intuitionistic fuzzy graph. Energy, Laplacian energy, and SLE of an intuitionistic fuzzy graph are evaluated using some examples, and their lower and upper bounds are derived.

Keywords: Energy of a graph; Fuzzy graph; Intuitionistic fuzzy graph; Laplacian energy; Seidel Laplacian energy

Introduction

Any system with a binary relation can be mathematically represented using graph theory (Harary, 1969; Bondy and Murthy, 1976), and a study by Zadeh from 1965 is where fuzzy sets first appeared (Zadeh, 1965). To express the phenomenon of uncertainty in real-world scenarios, Zadeh created the idea of a fuzzy subset of a set. Fuzzy graphs were first developed by Azriel

Rosenfeld in 1975. Graph theory is an important tool to address many real-world problems. The energy of a graph is one of the emerging concepts within graph theory (Gutman, 1978; Balakrishnan, 2004; Li *et al.*, 2012). Labelling and topological indices are also some of the main research concepts (Hemalatha and Somasundaram, 2022; Kavitha and Jayalalitha, 2023; Kavitha *et al.*, 2024), and a wide

¹ Department of Mathematics, Sri Eshwar College of Engineering, Coimbatore, 641202, India.
Email: sivaranjaniph.d2021@gmail.com*

² Department of Mathematics, Sree Saraswathi Thyagaraja College, Pollachi, Coimbatore, 642107, India.
Email: ovs3662@gmail.com

* Corresponding author

DOI: <https://doi.org/10.55766/sujst7197>

Suranaree J. Sci. Technol. 32(1):010350(1-14)

range of fuzzy applications that often involve them are epidemic modelling, optimisation of economic order quantities, and decision-making (Umapathy *et al.*, 2023; Jansirani *et al.*, 2024; Kuppusamy *et al.*, 2024). Other than graph theory, fuzzy and intuitionistic fuzzy concepts have also been widely applied in recent research works.

In the language of mathematics, a graph's energy is the sum of the absolute values of its adjacency matrix's eigenvalues. In the framework of spectral graph theory, this quantity is examined. Between mathematics and chemistry, this idea acts as a boundary. Many graphs have known energies. Nonetheless, we have decided to look at the energy of a graph that is produced on a given graph by various graph operations. Numerous concepts, such as graph energy, Seidel energy, Laplacian energy, Seidel Laplacian energy, distance energy, and many more, can be considered based on the notion of a matrix (Gutman and Zhou, 2006; Shigehalli and Betageri, 2015; Oboudi, 2016; Bhat and D'Souza, 2017; Ramane *et al.*, 2017; D'Souza *et al.*, 2022; Yalcin, 2022; Harishchandra *et al.*, 2023). Laplacian energy and graph energy have a high correlation with a number of vertex properties, including eigencentality, betweenness, and vertex degree. The strong association between graph energy, Laplacian energy, eigen centrality and, betweenness implies that techniques for determining eigen centrality and betweenness values can be developed using local vertex energy rather than a costly global computation spanning the entire network. Throughout this article, the number of vertices and edges of the graph G will be denoted by n and m , respectively. The study of the energy of a graph is extended to fuzzy graph, a hesitancy fuzzy graph, and is not limited to (Narayanan and Mathew, 2013; Reddy *et al.*, 2023). Consequently, intuitionistic fuzzy sets (IFS), a generalization of fuzzy sets, were proposed. IFS included the degree of hesitation known as the hesitation margin, which is defined as 1 minus the sum of the membership and non-membership degrees, respectively (Atanoso, 1986). In a short amount of time, Atanoso invented intuitionistic fuzzy sets (IFS), a novel kind of fuzzy set that is now employed in many branches of research and engineering and it is interesting as well as useful to problem solving. It is a fuzzy set extension where decision-makers define the elements' degree of membership and non-membership to the particular set.

Example 1.1: The group of students who receive scores of 0 or higher (out of 100) might be regarded as a discourse domain. Here are descriptions of the intuitionistic fuzzy set's membership, hesitation, and non-membership

degrees: If membership degrees are used to indicate low scores in student groups ranging from 0 to 50, then non-membership degrees can be used to indicate high scores in student groups 60 and above. Grades in the student group that fall between 51 and 59 might be categorised as either high or poor. These pupils (marks 51-59) can therefore be represented by hesitating degrees. The IFS theory has been used recently in a variety of fields, including medical diagnostics, logic programming, and decision-making issues. Intuitionistic fuzzy sets are the most widely used fuzzy set extension in the literature.

In 2006, Parvathi and Karunambigai defined intuitionistic fuzzy graphs based on IFS (Karunambigai and Parvathi, 2006) and to study more, refer to (Naggor and Shajitha, 2010; Akram and Akmal, 2016; Bhat and D'Souza, 2017; Ramane *et al.*, 2017). To study more concepts about the energy of an intuitionistic fuzzy graph and the energy of a Laplacian intuitionistic fuzzy graph, refer to (Praba and Chandrasekaran, 2014; Basha and Kartheek, 2015; Deepa and Chandrasekaran, 2018, Ramesh and Basha, 2020). In Basha and Kartheek, (2015), the author discussed the Laplacian energy of an intuitionistic fuzzy graph. Here are the results discussed for Seidel Laplacian energy.

The author, Sivaranjani *et al.*, extends the SLE of graphs to fuzzy graphs (Sivaranjani *et al.*, 2024). Here, the SLE of a graph is extended to an intuitionistic fuzzy graph. The majority of authors' current research focuses on solar energy and nanomaterial-based energy applications. Because nanotechnology produces inexpensive, clean energy, it has a significant impact on human life. The use of nanomaterials presents both significant benefits and challenges; potential solutions to these problems are also mentioned (Hussein, 2023; Divya *et al.*, 2024). Basic properties of SLE and its eigen values are discussed in some articles (Meenakshi and Lavanya, 2014; Askari, 2023; Askari *et al.*, 2024;).

The following is the article's schedule: A brief introduction, preliminary information, and clarification notes are provided in Sections 1 and 2. An analysis of an intuitionistic graph's energy is given in Section 3 along with some instances. An energy analysis of a Laplacian intuitionistic graph with examples is presented in Section 4. An analysis of a Seidel Laplacian intuitionistic graph's energy is provided with examples in Section 5. Finally, the lower and upper limitations on Seidel Laplacian energy are covered in Section 6. Additionally, Python programming code for calculating a graph's energy is provided in the Appendix section.

Preliminaries

Consider a graph ζ with “n” vertices and “m” edges. **Definition 2.1: Adjacency matrix $A(\zeta)$** (Ramane *et al.*, 2017).

An edge that is connected by two nodes (vertices) is called adjacent.

Generally, $A = [a_{ij}] = A(\zeta)$ is an adjacency matrix of ζ with ‘k’ rows and ‘l’ columns is defined by:

$$A = [a_{ij}] = \begin{cases} 1 & \text{if } (u_i, u_j) \in E \\ 0 & \text{if otherwise} \end{cases}$$

Definition 2.2: Energy of a graph $E(\zeta)$ (Gutman, 1978).

The sum of modulus of latent values is called as Energy of a graph where the roots of the expression $\det(eI - A(G)) = 0$ are known as latent (eigen) values of G .

Definition 2.3: Laplacian energy $LE(\zeta)$ (Gutman and Zhou, 2006).

The Laplacian matrix of ζ is defined as $L(\zeta) = D(\zeta) - A(\zeta)$. Let $e_1, e_2, \dots, e_n$ be its latent values. Then $LE(\zeta)$ is defined as $LE(\zeta) = \sum_{i=1}^n |e_i - \frac{2m}{n}|$ where $D(\zeta) = \text{diag}(d_1, d_2, d_3, \dots, d_n)$ be the diagonal matrix of vertex degrees.

Definition 2.4: Seidel matrix (Ramane *et al.*, 2017).

The seidel matrix of a graph is the real symmetric matrix $S(\zeta) = S_{ij}$, Where $S_{ij} = -1$, if the vertices are adjacent and $S_{ij} = 1$ if the vertices are not adjacent and $S_{ij} = 0$ if $i=j$.

Definition 2.5: Seidel Laplacian Matrix of $A(\zeta)$ (Ramane *et al.*, 2017).

The Seidel Laplacian matrix of a graph ζ as defined $SL(\zeta) = DS(\zeta) - S(\zeta)$.

Definition 2.6: Seidel Laplacian energy $SLE(\zeta)$ (Ramane *et al.*, 2017).

Let $E_1, E_2, \dots, E_n$ be the latent values of $SL(\zeta)$ and define SLE energy of a graph ζ as $SLE(\zeta) = \sum_{i=1}^n |E_i - \frac{n(n-1)-4m}{n}|$.

Definition 2.7: Fuzzy set: (Basha and Kartheek, 2015)

Let X be a nonempty set. A fuzzy set A in X is $A = \{x, \sigma_A(x)/x \in X\}$ which is characterized by a membership function $\sigma_A(x): X \rightarrow (0,1)$ and a fuzzy set satisfying the condition $\sigma_A(x) + \tau_A(x) = 1$ where $\tau_A(x) = 1 - \sigma_A(x)$ is called non-membership function.

Definition 2.8: Fuzzy graph (Basha and Kartheek, 2015).

A fuzzy graph with V as the underlying set is a pair of the function $\zeta = (\sigma, \tau)$ where $\sigma: V \rightarrow [0,1]$ is a fuzzy subset and $\tau: V * V \rightarrow [0,1]$ is a symmetric fuzzy relation on σ for all $u, v \in V$ such that $\tau(u, v) \leq \sigma(u) \cap \sigma(v)$.

Definition 2.9: Intuitionistic fuzzy set (Basha and Kartheek, 2015).

Let $\emptyset$ be a nonempty set. An intuitionistic fuzzy set A in $\emptyset$ is $A = \{x, \sigma_A(x), \tau_A(x)/x \in \emptyset\}$ with membership function $\sigma_A(x): \emptyset \rightarrow [0,1]$ and non-membership function $\tau_A(x): \emptyset \rightarrow [0,1]$ satisfying the following condition:

$$(i) \ 0 \leq \sigma_A(x) + \tau_A(x) \leq 1$$

$$(ii) \ 0 \leq \sigma_A(x) + \tau_A(x), \gamma_A(x) \leq 1$$

where $\gamma_A(x)$ is intuitionistic fuzzy index element “x” in A .

Definition 2.10 (Basha and Kartheek, 2015).

Let “u” be the vertex of the fuzzy graph $\zeta(\sigma, \mu)$. The degree of u is $d(u) = \sum_{uv \in E(\zeta)} \mu(uv)$.

The energy, Laplacian energy, and Seidel Laplacian energy of an intuitionistic fuzzy graphs are computed using examples in the following sections. Also, lower and upper bound of the Seidel Laplacian energy of an intuitionistic fuzzy graph are derived and verified through examples.

Remark 2.1 (Meenakshi and Lavanya, 2014): A graph has zero energy if and only if it is trivial. The energy of a nontrivial network is an even number if it is rational, according to the rational root theorem.

Remark 2.2 (Meenakshi and Lavanya, 2014): If a graph ζ is made up of (disconnected) components $\zeta_1, \zeta_2, \dots, \zeta_p$, then the energy of ζ is equal to $E(\zeta) = E(\zeta_1) + E(\zeta_2) + \dots + E(\zeta_p)$.

Energy of an Intuitionistic Fuzzy Graph

Definition 3.1: Intuitionistic fuzzy graph (IFG) (Karunambigai and Parvathi, 2006)

Intuitionistic fuzzy graph (IFG) is of the form $\zeta = (V, E, \sigma, \tau)$, where:

(i) $V = \{v_1, v_2, \dots, v_n\}$ such that $\sigma_1: V \rightarrow [0,1]$ and $\tau_1: V \rightarrow [0,1]$ denote the degrees of membership and non-membership of the element $v_i \in V$ respectively and $0 \leq \sigma_1(v_i) + \tau_1(v_i) \leq 1$, for every $v_i \in V$, $(i = 1, 2, \dots, n)$.

(ii) $E \subset V \times V$ where $\sigma_2: V \times V \rightarrow [0,1]$ are such that:

$$\sigma_2(v_i, v_j) \leq \min[\sigma_1(v_i), \sigma_1(v_j)]$$

$$\tau_2(v_i, v_j) \leq \max[\tau_1(v_i), \tau_1(v_j)]$$

And $0 \leq \sigma_2(v_i, v_j) + \tau_2(v_i, v_j) \leq 1$ for every $(v_i, v_j) \in E$.

Definition 3.2: Intuitionistic fuzzy adjacency matrix (Praba and Chandrasekaran, 2014).

An intuitionistic fuzzy graph is defined as the adjacency matrix of the corresponding intuitionistic fuzzy graph. That is for an intuitionistic fuzzy graph $\zeta = (V, E, \sigma, \tau)$, an intuitionistic fuzzy adjacency matrix is defined by $A = [a_{ij}] = [\sigma_{ij}, \tau_{ij}]$ where σ_{ij} represents the strength of the relationship between the vertices and τ_{ij} represents the strength of the non-relationship between the vertices.

Definition 3.3: Energy of an intuitionistic fuzzy graph (Praba and Chandrasekaran, 2014).

The energy of an intuitionistic fuzzy graph $\zeta = (V, E, \sigma, \tau)$ is $(\sum_{i=1}^n |\sigma_{ij}|, \sum_{i=1}^n |\tau_{ij}|)$.

Remark 3.1: Consider the adjacency matrix $A(IFG)$. The latent (eigen) values of $A(IFG)$ is defined as (E, e) where E is the set of latent values of $A(\sigma_{ij})$ and e is the set of latent value of $A(\tau_{ij})$.

Example 3.1: Consider the following graph ζ

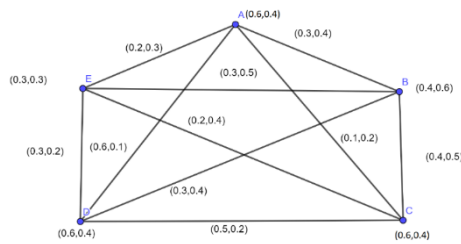


Figure 1. Intuitionistic fuzzy graph 1

This example 3.1 as shown in Figure. 1 satisfies condition (i) and (ii) in Definition 3.1. Therefore, it is an intuitionistic fuzzy graph.

Example 3.2: Consider the following graph ζ

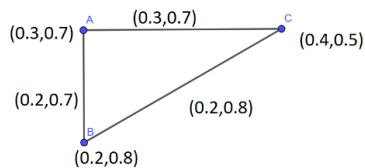


Figure 2. Intuitionistic fuzzy graph 2

This example 3.2 as shown in Figure. 2 satisfies condition (i) and (ii) in Definition 3.1. Therefore, it is an intuitionistic fuzzy graph.

Example 3.3: Consider the following graph ζ

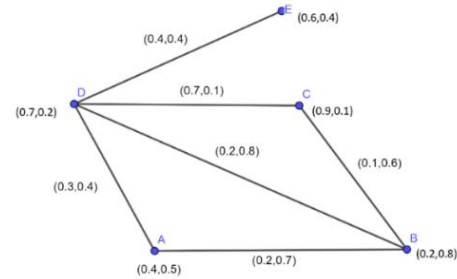


Figure 3. Intuitionistic fuzzy graph 3

This example 3.3 as shown in Figure. 3 satisfies condition (i) and (ii) in Definition 3.1. Therefore, it is an intuitionistic fuzzy graph. The adjacency matrix of an Example 3.1 is follows:

$$\begin{bmatrix} 0 & (0.3,0.4) & (0.1,0.2) & (0.6,0.1) & (0.2,0.3) \\ (0.3,0.4) & 0 & (0.4,0.5) & (0.3,0.4) & (0.3,0.5) \\ (0.1,0.2) & (0.4,0.5) & 0 & (0.5,0.2) & (0.6,0.1) \\ (0.6,0.1) & (0.3,0.4) & (0.5,0.2) & 0 & (0.3,0.2) \\ (0.2,0.3) & (0.3,0.5) & (0.6,0.1) & (0.3,0.2) & 0 \end{bmatrix}$$

The following two matrices containing membership and non- membership values which is the adjacency matrix of an intuitionistic fuzzy graph.

$A(IFG) = A(\sigma, \tau)$ where:

$$A(\sigma) = A(\sigma_{ij}) = \begin{bmatrix} 0 & 0.3 & 0.1 & 0.6 & 0.2 \\ 0.3 & 0 & 0.4 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0 & 0.5 & 0.6 \\ 0.6 & 0.3 & 0.5 & 0 & 0.3 \\ 0.2 & 0.3 & 0.6 & 0.3 & 0 \end{bmatrix}$$

Its corresponding characteristic equation is $-\lambda^5 + \frac{77}{50}\lambda^3 + \frac{411}{500}\lambda^2 + \frac{157}{2500}\lambda - \frac{363}{25000} = 0$, And its latent values are 10.6025, -3.6208, -2.0897, -1.2524, 0.6855. Its Energy is 18.2509.

$$\text{And } A(\tau) = A(\tau_{ij}) = \begin{bmatrix} 0 & 0.4 & 0.2 & 0.1 & 0.3 \\ 0.4 & 0 & 0.5 & 0.4 & 0.5 \\ 0.2 & 0.5 & 0 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.2 & 0 & 0.2 \\ 0.3 & 0.5 & 0.1 & 0.2 & 0 \end{bmatrix}$$

Its corresponding characteristic equation is $-\lambda^5 + \frac{21}{20}\lambda^3 + \frac{241}{500}\lambda^2 + \frac{153}{2500}\lambda + \frac{99}{50000} = 0$ and its latent values are -20.1141, -7.2625, -2.8968, -1.4556, 0.8199. Its energy is 32.5489.

(i) For Example 3.2,

The following two matrices containing membership and non- membership values which is the adjacency matrix of an intuitionistic fuzzy graph $A(IFG) = A(\sigma, \tau)$ where:

$$A(\sigma) = A(\sigma_{ij}) = \begin{bmatrix} 0 & 0.2 & 0 & 0.3 & 0 \\ 0.2 & 0 & 0.1 & 0.2 & 0 \\ 0 & 0.1 & 0 & 0.7 & 0 \\ 0.3 & 0.2 & 0.7 & 0 & 0.4 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

Its corresponding characteristic equation is $-\lambda^5 + \frac{83}{100}\lambda^3 + \frac{13}{250}\lambda^2 - \frac{201}{10000}\lambda = 0$, Latent values are 0, 0.02702, -0.09034, -0.87638, 0.93971 and Sum of the energy is

$$E_{\sigma}(IFG) = \sum_{i=1}^n |E_i| = 0.02702 + 0.09034 + 0.87638 + 0.93971 = 1.93345 \text{ and}$$

$$A(\tau) = A(\tau_{ij}) = \begin{bmatrix} 0 & 0.7 & 0 & 0.4 & 0 \\ 0.7 & 0 & 0.6 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.1 & 0 \\ 0.4 & 0.8 & 0.1 & 0 & 0.4 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

Its corresponding characteristic equation is $\lambda^5 + \frac{83}{50}\lambda^3 + \frac{68}{125}\lambda^2 - \frac{289}{10000}\lambda = 0$., Latent values are 0, 0.04653, -0.41212, -1.0584, 1.424, and Sum of the energy is $E_{\tau}(IFG) = \sum_{i=1}^n |E_i| = 0 + 0.04653 + 0.41212 + 1.0584 + 1.424 = 2.94105$.

(ii) The adjacency matrix of an Example 3.3 is as follows:

$$A(IFG) = \begin{bmatrix} 0 & (0.2,0.7) & (0.3,0.7) \\ (0.2,0.7) & 0 & (0.2,0.8) \\ (0.3,0.7) & (0.2,0.8) & 0 \end{bmatrix}$$

The following two matrices containing membership and non-membership values which is the adjacency matrix of an intuitionistic fuzzy graph

$$A(IFG) = A(\sigma, \tau) \text{ where, } A(\sigma_{ij}) = \begin{bmatrix} 0 & 0.2 & 0.3 \\ 0.2 & 0 & 0.2 \\ 0.3 & 0.2 & 0 \end{bmatrix}$$

Its corresponding characteristic equation is $-\lambda^3 + \frac{17}{100}\lambda + \frac{3}{125} = 0$ and its latent values are -0.3, -0.17016 and 0.47016. Its energy is 0.94032.

$$A(\tau_{ij}) = \begin{bmatrix} 0 & 0.7 & 0.7 \\ 0.7 & 0 & 0.8 \\ 0.7 & 0.8 & 0 \end{bmatrix}$$

Its corresponding characteristic equation is $-\lambda^3 + \frac{81}{50}\lambda + \frac{98}{125} = 0$ and its latent values are -0.8, -0.66771 and 1.4677. Its energy is 2.9354.

Graphical representation of energy values of membership and non-membership values is presented in Figures 4 and 5.

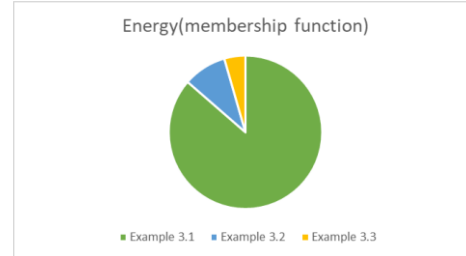


Figure 4. Graphical representation of energy values of membership values

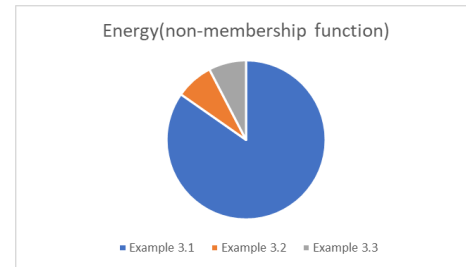


Figure 5. Graphical representation of energy values of non membership values

Comparison of energy values of an intuitionistic fuzzy graph represented in the Table 1 and Figure 6.

Table 1. Comparison of energy values of an intuitionistic fuzzy graph

Example	Energy-Membership function	Energy-Non membership function
3.1	18.2509	32.5489
3.2	1.93345	2.94105
3.3	0.94032	2.9354

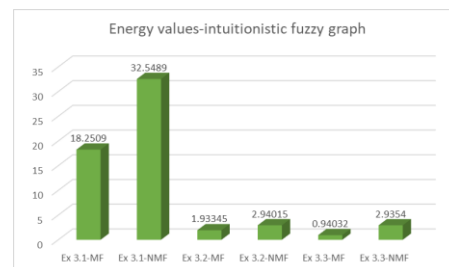


Figure 6. Comparison of energy values of an intuitionistic fuzzy graph

Summary report: 1

In this section, energy values of an intuitionistic fuzzy graph are evaluated for Example 3.1, 3.2 and 3.3 and its energy values for membership and non-membership values are graphically compared separately in Figure 4, 5 and graphically compared all together in Figure 6. Also energy values are tabulated in Table 1. (That is, In Figure 4, graphical representation of energy values of membership values of Example 3.1 (green color), 3.2 (blue color), and 3.3 (yellow color) is presented and in Figure 5, graphical representation of energy values of non membership values of Example 3.1 (green color), 3.2 (blue color), and 3.3 (yellow color) are presented).

Remark 3.2: Ex abbreviated as Example, MF as Membership Function and NMF as Non Membership Function.

Laplacian Energy of an Intuitionistic Fuzzy Graph

Definition 4.1: Laplacian matrix of an intuitionistic fuzzy graph (Basha and Kartheek, 2015). The Laplacian matrix of an intuitionistic fuzzy graph G is defined as $L(IFG) = D(IFG) - A(IFG)$. $L(IFG)$ also, can be written as $L(IFG) = (L(\sigma_{ij}), L(\tau_{ij}))$ where $D(IFG) = \text{diag}(d_1, d_2, d_3 \dots \dots d_n)$ be the diagonal matrix of vertex degrees.

Definition 4.2: Laplacian energy of an intuitionistic fuzzy graph (Basha and Kartheek, 2015) Consider ζ be an intuitionistic fuzzy graph with 'n' vertices and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots \dots \geq \lambda_n$ is the Laplacian latent values of IFG. The Laplacian energy of intuitionistic fuzzy graph is defined as $LE(\sigma(\zeta)) = \left| \lambda_i - 2 \frac{\sum_{i,j} \sigma_{ij}}{n} \right|$ Where $1 \leq i \leq j \leq n$ and $\delta_1 \geq \delta_2 \geq \dots \delta_n$ is the non membership Laplacian latent values of IFG. Then this energy is defined as $LE(\tau(\zeta)) = \left| \delta_i - 2 \frac{\sum_{i,j} \tau_{ij}}{n} \right|$ Where $1 \leq i \leq j \leq n$.

Remark 4.1: For membership function, the Laplacian matrix of an intuitionistic fuzzy graph ζ is defined as $L_\sigma(IFG) = D_\sigma(IFG) - A_\sigma(IFG)$

For non-membership function, Laplacian matrix of an intuitionistic fuzzy graph ζ is defined as $L_\tau(IFG) = D_\tau(IFG) - A_\tau(IFG)$

(i) For the Example 3.1, The Laplacian matrix of IFG of ζ

$$L(\sigma) = \begin{bmatrix} 1.2 & -0.3 & -0.1 & -0.6 & -0.2 \\ -0.3 & 1.3 & -0.4 & -0.3 & -0.3 \\ -0.1 & -0.4 & 1.6 & -0.5 & -0.6 \\ -0.6 & -0.3 & -0.5 & 1.7 & -0.3 \\ -0.2 & -0.3 & -0.6 & -0.3 & 1.4 \\ 1 & -0.4 & -0.2 & -0.1 & -0.3 \\ -0.4 & 1.8 & -0.5 & -0.4 & -0.5 \\ -0.2 & -0.5 & 1 & -0.2 & -0.1 \\ -0.1 & -0.4 & -0.2 & 0.9 & -0.2 \\ -0.3 & -0.5 & -0.1 & -0.2 & 1.1 \end{bmatrix} \text{ and } L(\tau) = \begin{bmatrix} 1.2 & -0.3 & -0.1 & -0.6 & -0.2 \\ -0.3 & 1.3 & -0.4 & -0.3 & -0.3 \\ -0.1 & -0.4 & 1.6 & -0.5 & -0.6 \\ -0.6 & -0.3 & -0.5 & 1.7 & -0.3 \\ -0.2 & -0.3 & -0.6 & -0.3 & 1.4 \\ 1 & -0.4 & -0.2 & -0.1 & -0.3 \\ -0.4 & 1.8 & -0.5 & -0.4 & -0.5 \\ -0.2 & -0.5 & 1 & -0.2 & -0.1 \\ -0.1 & -0.4 & -0.2 & 0.9 & -0.2 \\ -0.3 & -0.5 & -0.1 & -0.2 & 1.1 \end{bmatrix}$$

The Laplacian energy of intuitionistic fuzzy graph $\zeta = (V, E, \sigma, \tau)$ is defined as $LE[\sigma(\zeta), \tau(\zeta)]$ For σ : characteristic polynomial is $\lambda(-2000\lambda^4 + 14400\lambda^3 - 38220\lambda^2 + 44284\lambda - 18883)$ Its latent values are 0, 2.3497, 1.9748, 1.6177, 1.2578 For τ : characteristic polynomial is $\lambda(-2000\lambda^4 + 11600\lambda^3 - 24280\lambda^2 + 21920\lambda - 7243)$ Its latent values are 0, 2.2638, 1.3805, 1.1320, and 1.0236.

$$LE(\sigma(\zeta)) = \left| 0 - 2\left(\frac{3.2}{5}\right) \right| + \left| 2.3497 - 2\left(\frac{3.2}{5}\right) \right| + \left| 1.9748 - 2\left(\frac{3.2}{5}\right) \right| + \left| 1.6177 - 2\left(\frac{3.2}{5}\right) \right| + \left| 1.2578 - 2\left(\frac{3.2}{5}\right) \right| = 1.28 + 1.0697 + 0.6948 + 0.3377 + 0.0222 = 3.4044$$

$$LE(\tau(\zeta)) = \left| 0 - 2\left(\frac{3.2}{5}\right) \right| + \left| 2.2638 - 2\left(\frac{3.2}{5}\right) \right| + \left| 1.3805 - 2\left(\frac{3.2}{5}\right) \right| + \left| 1.1320 - 2\left(\frac{3.2}{5}\right) \right| + \left| 1.0236 - 2\left(\frac{3.2}{5}\right) \right| = 1.28 + 0.9838 + 0.1005 + 0.148 + 0.2564 = 2.7687$$

Here,

$$\sum_{i,j} \sigma(u_i, u_j) = 0.2 + 0.3 + 0.4 + 0.5 + 0.3 + 0.2 + 0.3 + 0.1 + 0.6 + 0.3 = 3.2 \text{ and } \sum_{i,j} \tau(u_i, u_j) = 0.3 + 0.4 + 0.5 + 0.2 + 0.2 + 0.4 + 0.5 + 0.2 + 0.4 + 0.1 = 3.2$$

Therefore, the Laplacian energy of intuitionistic fuzzy graph ζ is [3.4044, 2.7687].

(ii) For Example 3.2, The following is the Laplacian matrix of IFG of ζ

$$L(\sigma) = \begin{bmatrix} 0.5 & -0.2 & 0 & -0.3 & 0 \\ -0.2 & 0.5 & -0.1 & -0.2 & 0 \\ 0 & -0.1 & 0.8 & -0.7 & 0 \\ -0.3 & -0.2 & -0.7 & 1.6 & -0.4 \\ 0 & 0 & 0 & -0.4 & 0.4 \end{bmatrix} \text{ and its}$$

corresponding characteristic equation is $-500\lambda^5 + 1900\lambda^4 - 2230\lambda^3 + 1034\lambda^2 - 163\lambda = 0$. Its latent values are 0, 2.8265, 1.6405, 1.4061, 0.4705

$$L(\tau) = \begin{bmatrix} 1.1 & -0.7 & 0 & -0.4 & 0 \\ -0.7 & 2.1 & -0.6 & -0.8 & 0 \\ 0 & -0.6 & 0.7 & -0.1 & 0 \\ -0.4 & -0.8 & -0.1 & 1.7 & -0.4 \\ 0 & 0 & 0 & -0.4 & 0.4 \end{bmatrix} \text{ Its}$$

corresponding characteristic equation is $-250\lambda^5 +$

$1500\lambda^4 - 2900\lambda^3 + 2044\lambda^2 - 439\lambda = 0$. Its latent values are 0, 2.5684, 1.2124, 0.5305, 0.3447.

Here, $\sum_{i,j} \sigma(u_i, u_j) = 0.4 + 0.7 + 0.1 + 0.2 + 0.2 + 0.3 = 1.9$ and $\sum_{i,j} \tau(u_i, u_j) = 0.4 + 0.1 + 0.6 + 0.8 + 0.7 + 0.4 = 3$. $LE(\sigma(\zeta)) = \left|0 - 2\left(\frac{1.9}{5}\right)\right| + \left|2.8265 - 2\left(\frac{1.9}{5}\right)\right| + \left|1.6405 - 2\left(\frac{1.9}{5}\right)\right| + \left|1.4061 - 2\left(\frac{1.9}{5}\right)\right| + \left|0.4705 - 2\left(\frac{1.9}{5}\right)\right| = 0.76 + 2.0665 + 0.8805 + 0.6461 + 0.2895 = 4.6426$.

$LE(\tau(\zeta)) = \left|0 - 2\left(\frac{3}{5}\right)\right| + \left|2.5684 - 2\left(\frac{3}{5}\right)\right| + \left|1.2124 - 2\left(\frac{3}{5}\right)\right| + \left|0.5305 - 2\left(\frac{3}{5}\right)\right| + \left|0.3447 - 2\left(\frac{3}{5}\right)\right| = 1.2 + 1.3684 + 0.0124 + 0.6695 + 0.8553 = 4.1056$.

Therefore, the Laplacian energy of intuitionistic fuzzy graph ζ is [4.6426, 4.1056].

- (iii) For Example 3.3, for membership function, Laplacian matrix of an intuitionistic fuzzy graph ζ is defined as:

$$L_{\sigma}(IFG) = D_{\sigma}(IFG) - A_{\sigma}(IFG) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} - \begin{bmatrix} 0 & 0.2 & 0.3 \\ 0.2 & 0 & 0.2 \\ 0.3 & 0.2 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.2 & -0.3 \\ -0.2 & 0.4 & -0.2 \\ -0.3 & -0.2 & 0.5 \end{bmatrix}$$

Its corresponding characteristic equation is $-25\lambda^3 + 35\lambda^2 - 12\lambda = 0$. and its latent values are 0, 0.6 and 0.8. Then this energy is defined as:

$$LE(\sigma(\zeta)) = \left| \lambda_i - 2 \frac{\sum_{i,j} \sigma_{ij}}{n} \right|$$

$$LE(\sigma(\zeta)) = \left|0 - 2\left(\frac{0.7}{3}\right)\right| + \left|0.6 - 2\left(\frac{0.7}{3}\right)\right| + \left|0.8 - 2\left(\frac{0.7}{3}\right)\right| = 0.4667 + 0.067 + 0.3333 = 0.867$$

For non-membership function, Laplacian matrix of an intuitionistic fuzzy graph ζ is defined as:

$$L_{\tau}(IFG) = D_{\tau}(IFG) - A_{\tau}(IFG) = \begin{bmatrix} 1.4 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} - \begin{bmatrix} 0 & 0.7 & 0.7 \\ 0.7 & 0 & 0.8 \\ 0.7 & 0.8 & 0 \end{bmatrix} = \begin{bmatrix} 1.4 & -0.7 & -0.7 \\ -0.7 & 1.5 & -0.8 \\ -0.7 & -0.8 & 1.5 \end{bmatrix}$$

Its corresponding characteristic equation is $-\lambda^3 + \frac{22}{5}\lambda^2 - \frac{483}{100}\lambda = 0$, And its latent values are 0, 2.1 and 2.3. Then this energy is defined as:

$$LE(\tau(\zeta)) = \left| \delta_i - 2 \frac{\sum_{i,j} \tau_{ij}}{n} \right| LE(\tau(\zeta)) = \left|0 - 2\left(\frac{2.2}{3}\right)\right| + \left|2.1 - 2\left(\frac{2.2}{3}\right)\right| + \left|2.3 - 2\left(\frac{2.2}{3}\right)\right| = 1.4667 + 0.6333 + 0.8333 = 2.9333$$

Graphical representation of Laplacian energy values of membership and non-membership values is presented in Figures 7 and 8.

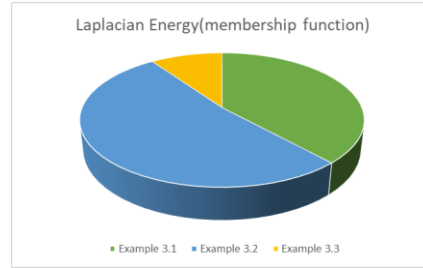


Figure 7. Graphical representation of Laplacian energy values of membership values

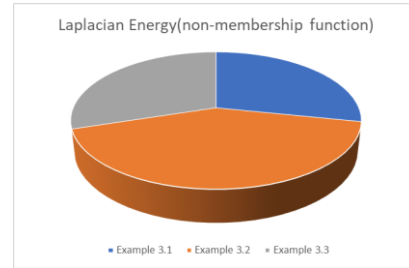


Figure 8. Graphical representation of Laplacian energy values of non membership values

Comparison of Laplacian energy values of an intuitionistic fuzzy graph represented in the Table 2 and Figure 9.

Table 2. Comparison of Laplacian energy values of an intuitionistic fuzzy graph

Example	Laplacian Energy-Membership function	Laplacian Energy-Non membership function
3.1	3.4044	2.7687
3.2	4.6426	4.1056
3.3	0.867	2.9333

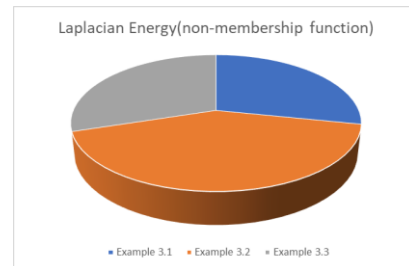


Figure 9. Comparison of Laplacian energy values of an intuitionistic fuzzy graph

In this section, Laplacian energy values are evaluated for Example 3.1, 3.2 and 3.3 and Laplacian energy values of membership and non-membership values are graphically compared separately in Figures 7 and 8 and graphically compared all together in Figure 9. Also Laplacian energy values are tabulated in Table 2. (That is, In Figure 7, graphical representation of Laplacian energy values of membership values of Example 3.1 (green color), 3.2 (blue color), and 3.3 (yellow color) and in Figure 8, graphical representation of Laplacian energy values of non membership values of Example 3.1 (blue color), 3.2 (brown color), and 3.3 (grey color) are presented).

Seidel Laplacian Energy (SLE) of an Intuitionistic Fuzzy Graph

Definition 5.1: Seidel Laplacian matrix of an intuitionistic fuzzy graph

The Seidel Laplacian matrix of an intuitionistic fuzzy graph G is defined as $SL(IFG) = DS(IFG) - S(IFG)$ where $DS(IFG) = \text{diag}(n-1-2d_1, n-1-2d_2, \dots, n-1-2d_n)$.

Definition 5.2: Seidel Laplacian polynomial

Consider the intuitionistic fuzzy SL matrix of IFG . The Seidel Laplacian polynomial is defined as $\det(\Delta I - SL(IFG))$.

The roots of the above expression are the intuitionistic fuzzy Seidel Laplacian eigen(latent) values of IFG .

Definition 5.3: Seidel Laplacian energy of an intuitionistic fuzzy graph

Let $E_1, E_2, E_3, \dots, E_n$ and $e_1, e_2, e_3, \dots, e_n$ be the Latent values of $SL_\sigma(IFG)$ and $SL_\tau(IFG)$ respectively. Then Energy of $SL_\sigma(IFG) = SLE_\sigma(IFG) = \sum_{i=1}^n \left| E_i - \frac{n(n-1)-4\sum_{i,j} \sigma(u_i u_j)}{n} \right|$ and Energy of $SL_\tau(IFG) = SLE_\tau(IFG) = \sum_{i=1}^n \left| e_i - \frac{n(n-1)-4\sum_{i,j} \tau(u_i u_j)}{n} \right|$.

The Seidel Laplacian energy of intuitionistic fuzzy graph is $\zeta = (SLE_\sigma(IFG), SLE_\tau(IFG))$.

Remark 5.1: For membership function, Seidel Laplacian matrix of an intuitionistic fuzzy graph ζ is defined as:

$$SL_\sigma(IFG) = DS_\sigma(IFG) - S(IFG)$$

where $DS_\sigma(IFG) = \text{diag}(n-1-2d_1, n-1-2d_2, \dots, n-1-2d_n)$.

For non-membership function, Seidel Laplacian matrix of an intuitionistic fuzzy graph ζ is defined as:

$$SL_\tau(IFG) = DS_\tau(IFG) - S(IFG)$$

where $DS_\tau(IFG) = \text{diag}(n-1-2d_1, n-1-2d_2, \dots, n-1-2d_n)$.

(i) For Example 3.1,

$$S(IFG) = \begin{bmatrix} 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix} \text{ And}$$

$$DS(IFG) = \text{diag}(n-1-2d_1, n-1-2d_2, \dots, n-1-2d_n)$$

For membership function,

$d_1 = 0.8, d_2 = 1.3, d_3 = 1.6, d_4 = 1.7, d_5 = 1.4$ and for non-membership function, $d_1 = 1.3, d_2 = 1.6, d_3 = 1.3, d_4 = 0.9, d_5 = 1.1$

$$DS_\sigma(IFG) = \begin{bmatrix} 2.4 & 0 & 0 & 0 & 0 \\ 0 & 1.4 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 1.2 \end{bmatrix} \text{ and}$$

$$DS_\tau(IFG) = \begin{bmatrix} 1.4 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 1.4 & 0 & 0 \\ 0 & 0 & 0 & 2.2 & 0 \\ 0 & 0 & 0 & 0 & 1.8 \end{bmatrix}$$

Therefore

$$SL_\sigma(IFG) = DS_\sigma(IFG) - S(IFG) = \begin{bmatrix} 2.4 & 1 & 1 & 1 & 1 \\ 1 & 1.4 & 1 & 1 & 1 \\ 1 & 1 & 0.8 & 1 & 1 \\ 1 & 1 & 1 & 0.6 & 1 \\ 1 & 1 & 1 & 1 & 1.2 \end{bmatrix}$$

Its characteristic equation is $-\lambda^5 + \frac{32}{5}\lambda^4 - \frac{27}{5}\lambda^3 - \frac{22}{25}\lambda^2 + \frac{346}{625}\lambda + \frac{48}{3125} = 0$.

$$SL_\tau(IFG) = DS_\tau(IFG) - S(IFG) = \begin{bmatrix} 1.4 & 1 & 1 & 1 & 1 \\ 1 & 0.8 & 1 & 1 & 1 \\ 1 & 1 & 1.4 & 1 & 1 \\ 1 & 1 & 1 & 2.2 & 1 \\ 1 & 1 & 1 & 1 & 1.8 \end{bmatrix}$$

Its characteristic equation is

$$\frac{(5\lambda - 2)(-625\lambda^4 + 4500\lambda^3 - 6050\lambda^2 + 1970\lambda + 148)}{3125} = 0.$$

For membership function, E_i values are -0.0268, -0.32308, 0.30731, 1.07569, 5.36688 and for non-membership function, e_i values are 0.4, 5.5632, 1.0532, 0.6462, -0.0625.

Also $n(n-1) = 5$.

Energy of $SL_\sigma(IFG) = |-0.0268 - 1.44| + |-0.32308 - 1.44| + |0.30731 - 1.44| + |1.07569 - 1.44| + |5.36688 - 1.44| = 1.4668 + 1.76308 + 1.13269 + 0.36431 + 3.92688 = 8.65376$. and Energy of $SL_\tau(IFG) = |0.4 - 1.44| + |5.5632 - 1.44| + |1.0532 - 1.44| + |0.6462 - 1.44| + |-0.0625 - 1.44| = 1.04 + 4.1232 + 0.3868 + 0.7938 + 1.5025 = 7.8463$.

The SLE of an intuitionistic fuzzy graph ζ is $(SLE_\sigma(IFG), SLE_\tau(IFG)) = (8.65376, 7.8463)$.

(ii) For Example 3.2

$$S(IFG) = \begin{bmatrix} 0 & -1 & 1 & -1 & 1 \\ -1 & 0 & -1 & -1 & 1 \\ 1 & -1 & 0 & -1 & 1 \\ -1 & -1 & -1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 0 \end{bmatrix}$$

and $DS(IFG) = \text{diag}(n-1-2d_1, n-1-2d_2, \dots, n-1-2d_n)$

$$DS_\sigma(IFG) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2.4 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0 & 3.2 \end{bmatrix} \text{ and } DS_\tau(IFG) = \begin{bmatrix} 1.8 & 0 & 0 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 \\ 0 & 0 & 2.6 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0 & 3.2 \end{bmatrix}$$

Therefore, $SL_\sigma(IFG) = DS_\sigma(IFG) - S(IFG)$ latent values are 0, 0.02702, -0.09034, -0.87638, 0.93971

$$SL_\tau(IFG) = DS_\tau(IFG) - S(IFG)$$

$$= \begin{bmatrix} 3 & 1 & -1 & 1 & -1 \\ 1 & 3 & 1 & 1 & -1 \\ -1 & 1 & 2.4 & 1 & -1 \\ 1 & 1 & 1 & 0.8 & 1 \\ -1 & -1 & -1 & 1 & 3.2 \end{bmatrix}$$

Its characteristic equation is $-\lambda^5 + \frac{62}{5}\lambda^4 - \frac{1239}{25}\lambda^3 + \frac{7288}{125}\lambda^2 + \frac{4922}{125}\lambda - \frac{8016}{125} = 0$.

Latent values are -1.0730, 0.9147, 0.3165, 0.2696, 0.1862.

$$SL_\tau(IFG) = DS_\tau(IFG) - S(IFG)$$

$$= \begin{bmatrix} 1.8 & 1 & -1 & 1 & -1 \\ 1 & 0.8 & 1 & 1 & -1 \\ -1 & 1 & 2.6 & 1 & -1 \\ 1 & 1 & 1 & 0.6 & 1 \\ -1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

Its characteristic equation is $-\lambda^5 + \frac{29}{5}\lambda^4 - \frac{33}{25}\lambda^3 - \frac{3767}{125}\lambda^2 + \frac{13196}{625}\lambda + \frac{1732}{125} = 0$.

Latent values are -2.3857, -0.4794, 0.7622, 0.3215, 0.2576.

Energy of $SL_\sigma(IFG) = |-1.0730 - 2.48| + |0.9147 - 2.48| + |0.3165 - 2.48| + |0.2696 - 2.48| + |0.1862 - 2.48| = 3.553 + 1.5653 + 2.1635 + 2.2114 + 2.2938 = 11.787$. and Energy of $SL_\tau(IFG) = |-2.3857 - 1.6| + |-0.4794 - 1.6| + |0.7622 - 1.6| + |0.3215 - 1.6| + |0.2576 - 1.6| = 3.9857 + 2.0794 + 0.8378 + 1.2785 + 1.3424 = 9.5238$.

The SLE of an intuitionistic fuzzy graph ζ is $(SLE_\sigma(IFG), SLE_\tau(IFG)) = (11.787, 9.5238)$.

(iii) For Example 3.3,

$$d_1 = 0.5, d_2 = 0.5, d_3 = 0.6$$

$$DS_\sigma(IFG) = \text{diag}(3-1-2(0.5), 3-1-2(0.4), 3-1-2(0.5)) = \text{diag}(1, 1.2, 1) =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and for non-membership function,}$$

$$DS_\tau(IFG) = \text{diag}(n-1-2d_1, n-1-2d_2, \dots, n-1-2d_n) \quad d_1 = 1.4, d_2 = 1.5, d_3 = 1.5 \quad DS_\tau(IFG) = \text{diag}(3-1-2(1.4), 3-1-2(1.5), 3-1-2(1.5)) = \text{diag}(-0.8, -1, -1) =$$

$$\begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } S(IFG) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$SL_{\sigma}(IFG) = DS_{\sigma}(IFG) - S(IFG) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Its corresponding characteristic equation is $-5\lambda^3 + 16\lambda^2 - 2\lambda = 0$ and its latent values are 0, 0.13031 and 3.06969.

$$SL_{\tau}(IFG) = DS_{\tau}(IFG) - S(IFG) = \begin{bmatrix} -0.8 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -0.8 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Its corresponding characteristic equation is $-\lambda^3 - \frac{14}{5}\lambda^2 + \frac{2}{5}\lambda + 4 = 0$ and its latent values are -2, -1.8697 and 1.0697. Energy of $SL_{\sigma}(IFG) = |0 - 0.8| + |0.13031 - 0.8| + |3.06969 - 0.8| = 0.8 + 0.66969 + 2.26969 = 3.73938$. and Energy of $SL_{\tau}(IFG) = |-2 + 0.9333| + |-1.8697 + 0.9333| + |1.0697 + 0.9333| = 1.0667 + 0.9364 + 2.003 = 4.0061$.

The SLE of an intuitionistic fuzzy graph ζ is $(SLE_{\sigma}(IFG), SLE_{\tau}(IFG)) = (3.73938, 4.0061)$.

Graphical representation of Seidel Laplacian energy values of membership and non-membership values is presented in Figures 10 and 11.

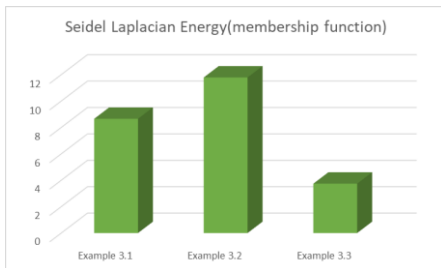


Figure 10. Comparison of Laplacian energy values of an intuitionistic fuzzy graph

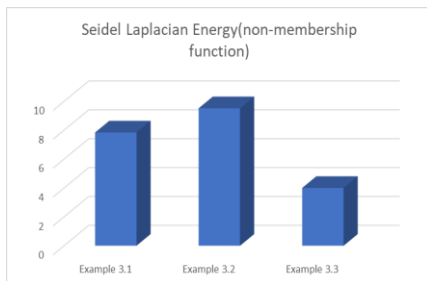


Figure 11. Graphical representation of Seidel Laplacian energy values of non membership values

Comparison of Seidel Laplacian energy values of an intuitionistic fuzzy graph represented in the Table 3 and Figure 12.

Table 3. Comparison of Seidel Laplacian energy values of an intuitionistic fuzzy graph

Example	Seidel Laplacian Energy-Membership function	Seidel Laplacian Energy-Non Membership function
3.1	8.65376	7.8463
3.2	11.787	9.5238
3.3	3.73938	4.0061

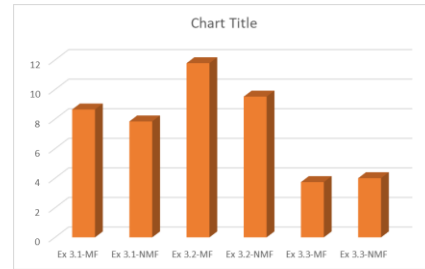


Figure 12. Comparison of Seidel Laplacian energy values of an intuitionistic fuzzy graph

Summary report: 3

In this section, Seidel Laplacian energy values are evaluated for Example 3.1, 3.2 and 3.3 and Seidel Laplacian energy values of membership and non-membership values are graphically compared separately in Figures 10 and 11 and graphically compared all together in Figure 12. Also Seidel Laplacian energy values are tabulated in Table 3. (That is, In Figure 10, graphical representation of Seidel Laplacian energy values of membership values of Example 3.1, 3.2 and 3.3 (green color) and in Figure 11, graphical representation of Seidel Laplacian energy values of non membership values of Example 3.1, 3.2 and 3.3 (blue color) are presented)

Lower and Upper bound of Seidel Laplacian Energy of an Intuitionistic fuzzy graph

Theorem 6.1:

Consider an intuitionistic fuzzy graph ζ with “n” vertices and “m” edges and $SL(IFG) = SL(\sigma, \tau)$ where $\sigma = \sigma_{ij}$ and $\tau = \tau_{ij}$ be an intuitionistic Seidel Laplacian matrix of ζ . Then

$$(i) \quad \frac{(SLE(\sigma(\zeta)))^2}{n^2} - 2 \sum_{i,j} \sigma^2 \leq \sum_{i=1}^n (D_{\sigma}(u) - 2 \frac{\sum_{i,j} \sigma}{n})^2$$

$$(ii) \frac{(SLE(\tau(\zeta)))^2}{n^2} - 2 \sum_{i,j} \tau^2 \leq \sum_{i=1}^n \left(D_\tau(u) - 2 \frac{\sum_{i,j} \tau}{n} \right)^2$$

$$i \leq i < j \leq n, \quad u = u_{ij}$$

Proof:

Using Cauchy - Schwarz inequality,

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right)$$

Assume $a_i = 1$ and $b_i = |\Delta_i|$, Then $(\sum_{i=1}^n |\Delta_i|)^2 \leq (\sum_{i=1}^n 1^2) (\sum_{i=1}^n |\Delta_i|^2) = n (\sum_{i=1}^n |\Delta_i|^2) (\sum_{i=1}^n |\Delta_i|) \leq \sqrt{n (\sum_{i=1}^n |\Delta_i|^2)}$

$$SLE(\sigma(\zeta)) \leq \sqrt{n \left(\sum_{i=1}^n |\Delta_i|^2 \right)} = \sqrt{2\theta}$$

Where, $\theta = \sum_{i,j} \sigma^2 + \frac{1}{2} \sum_{i=1}^n (D_\sigma(u_i) - 2 \frac{\sum_{i,j} \sigma}{n})^2$

$$SLE(\sigma(\zeta)) \leq \sqrt{2 \sum_{i,j} \sigma^2 + \frac{1}{2} \sum_{i=1}^n (D_\sigma(u_i) - 2 \frac{\sum_{i,j} \sigma}{n})^2}$$

Therefore, $\frac{(SLE(\sigma(\zeta)))^2}{n^2} - 2 \sum_{i,j} \sigma^2 \leq \sum_{i=1}^n (D_\sigma(u) - 2 \frac{\sum_{i,j} \sigma}{n})^2$ and $\frac{(SLE(\tau(\zeta)))^2}{n^2} - 2 \sum_{i,j} \tau^2 \leq \sum_{i=1}^n (D_\tau(u) - 2 \frac{\sum_{i,j} \tau}{n})^2$.

Checking the result with Example 3.1, For membership function (i) result holds.

$$8.65376 \leq 5 \sqrt{2.44 + (1.1664 + 0.9604 + 0.7744 + 0.6084 + 0.9604 + 1.1664 + 0.9604 + 1.3924 + 0.4624 + 0.904)} \leq 5\sqrt{11.7956} = 17.17.$$

and for non-membership function (ii) result holds.

$$7.8463 \leq 5 \sqrt{2.4 + (0.9604 + 0.7744 + 0.6084 + 1.1664 + 1.1664 + 0.7744 + 0.6084 + 1.1664 + 0.7744 + 1.3924)} \leq 5\sqrt{11.792} = 17.15.$$

Checking the result with Example 3.2, For membership function (i) result holds.

$$11.706 \leq 5 \sqrt{1.66 + (0.1296 + 0.0036 + 0.4356 + 2.8315 + 0.3136 + 0.3136)} \leq 5\sqrt{5.6875} = 11.9242.$$

and for non-membership function (ii) result holds.

$$9.5238 \leq 5 \sqrt{3.64 + 0.64 + 1.21 + 0.36 + 0.16 + 0.64 + 0.25} \leq 5\sqrt{6.9} = 13.1339.$$

Checking the result with Example 3.3, For membership function (i) result holds.

$$3.73938 \leq 3 \sqrt{1.34 + (0.25 + 0.25 + 0.16)} \leq 3\sqrt{2} = 4.2426.$$

and for non-membership function (ii) result holds.

$$4.0061 \leq 3 \sqrt{3.24 + 2.25 + 2.25 + 1.96} \leq 3\sqrt{9.7} = 9.3434.$$

Theorem 6.2:

Consider an intuitionistic fuzzy graph ζ “n” vertices and “m” edges and $SL_E(\text{IFG}) = SL(\sigma, \tau)$ where $\sigma = \sigma_{ij}$ and $\tau = \tau_{ij}$ be an intuitionistic Seidel Laplacian matrix of ζ .

Then

$$(i) \left(SLE(\sigma(\zeta)) \right)^2 \geq 4 \left(\sum_{i,j} \sigma^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\sigma(u) - 2 \frac{\sum_{i,j} \sigma}{n} \right)^2 \right)$$

$$(ii) \left(SLE(\tau(\zeta)) \right)^2 \geq 4 \left(\sum_{i,j} \tau^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\tau(u) - 2 \frac{\sum_{i,j} \tau}{n} \right)^2 \right)$$

$$i \leq i < j \leq n, u = u_{ij}$$

Proof:

Consider $Sle(\zeta)^2 = (\sum_{i=1}^n |\Delta_i|)^2 = \sum_{i=1}^n |\Delta_i|^2 + 2 \sum_{1 \leq i < j < n} |\Delta_i| |\Delta_j| \geq \sum_{i=1}^n |\Delta_i|^2 + 2 \sum_{i < j} |\Delta_i \Delta_j|$

Where, $\theta = \sum_{i,j} \sigma^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\sigma(u_i) - 2 \frac{\sum_{i,j} \sigma}{n} \right)^2$

Therefore $\left(SLE(\sigma(\zeta)) \right)^2 \geq 4 \left(\sum_{i,j} \sigma^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\sigma(u) - 2 \frac{\sum_{i,j} \sigma}{n} \right)^2 \right)$ and similarly, $\left(SLE(\tau(\zeta)) \right)^2 \geq 4 \left(\sum_{i,j} \tau^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\tau(u) - 2 \frac{\sum_{i,j} \tau}{n} \right)^2 \right)$.

Checking the result with Example 3.1, For membership function (i) result holds.

$$8.65376 \geq 2 \sqrt{1.22 + \frac{9.3556}{2}} = 2\sqrt{5.8978} = 4.85708$$

and for non-membership function (ii) result holds.

$$7.8463 \geq 2 \sqrt{1.2 + \frac{9.392}{2}} = 2\sqrt{5.896} = 4.8563.$$

Checking the result with Example 3.2, For membership function (i) result holds.

$$11.706 \geq 2 \sqrt{0.83 + \frac{1.4076}{2}} = 2\sqrt{1.5338} = 2.4769.$$

and for non-membership function (ii) result holds.

$$9.5238 \geq 2 \sqrt{1.82 + \frac{3.26}{2}} = 2\sqrt{3.45} = 3.7148.$$

Checking the result with Example 3.3, For membership function (i) result holds.

$$3.73938 \geq 2 \sqrt{0.17 + \frac{0.66}{2}} = 2\sqrt{0.50} = 1.4142$$

and for non-membership function (ii) result holds.

$$4.0061 \geq 2 \sqrt{2.44 + \frac{3.46}{2}} = 2\sqrt{3.97} = 3.9848.$$

Theorem 6.3:

Let ζ be an intuitionistic fuzzy graph with “n” vertices and “m” and $SL_E(IFG) = SL(\sigma, \tau)$ where $\sigma = \sigma_{ij}$ and $\tau = \tau_{ij}$ be an intuitionistic Seidel Laplacian matrix of ζ . Then,

$$(i) \quad SLE(\sigma(\zeta)) \leq \varepsilon_i + \sqrt{2 \sum_{i,j} \sigma^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\sigma(u_i) - 2 \frac{\sum_{i,j} \sigma}{n} \right)^2} - \varepsilon_i^2$$

$$(ii) \quad SLE(\tau(\zeta)) \leq \varepsilon_i + \sqrt{2 \sum_{i,j} \tau^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\tau(u_i) - 2 \frac{\sum_{i,j} \tau}{n} \right)^2} - \varepsilon_i^2$$

Proof:

Apply Cauchy Schwarz inequality we get,

$$\left| \sum_{i=2}^n \varepsilon_i \right|^2 \leq (n-1)(|\varepsilon_2|^2 + |\varepsilon_3|^2 + \dots + |\varepsilon_n|^2)$$

where $SLE(\sigma(G)) - \varepsilon_i \leq \sqrt{(n-1)(2\theta - \varepsilon_i^2)}$
Therefore,

$$SLE(\sigma(\zeta)) \leq \varepsilon_i + \sqrt{2 \sum_{i,j} \sigma^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\sigma(u_i) - 2 \frac{\sum_{i,j} \sigma}{n} \right)^2} - \varepsilon_i^2$$

Similarly,

$$SLE(\tau(\zeta)) \leq \varepsilon_i + \sqrt{2 \sum_{i,j} \tau^2 + \frac{1}{2} \sum_{i=1}^n \left(D_\tau(u_i) - 2 \frac{\sum_{i,j} \tau}{n} \right)^2} - \varepsilon_i^2.$$

Conclusion

In the graph theory research area, the study of the energy of a graph is extended to fuzzy graphs, hesitancy fuzzy graphs, picture fuzzy graphs, and intuitionistic fuzzy graphs and is not limited to. Here, the Seidel Laplacian energy (SLE) for an intuitionistic fuzzy graph is defined. Energy, Laplacian energy, and SLE of an intuitionistic fuzzy graph are evaluated using some examples, and lower and upper bounds are derived. In further study, the nature of Seidel Laplacian energy will be analysed for bipolar graphs, bipolar fuzzy graphs, and bipolar intuitionistic fuzzy graphs. Also, lower bounds and upper bounds will be evaluated. Energy device performance has been significantly improved by nanotechnology, which has also enhanced functional nanomaterials. Future research will examine how the characteristics of nanomaterials affect energy applications, such as energy production, conversion, efficient storage, and transportation

Conflict of Interest

All the authors hereby declare that they do not have any conflict of interests.

Appendix

Python programming code for energy of a graph

The following python code helps to calculate energy of a graph for readers.

For any $n \times n$ matrix, the characteristic polynomial, latent(eigen) values and sum of the modulus of the latent values were calculated using the following python programming code: In this paper the code is used in Example 3.1, 3.2 and 3.3.

```
import numpy as np
import sympy as sp
```

```
def create_matrix(n):
    matrix = np.zeros((n, n))
    for i in range(n):
        for j in range(n):
            matrix[i, j] = float(input(f"Enter element at
position ({i+1}, {j+1}): "))
    return matrix

def characteristic_polynomial(matrix):
    X = sp.Symbol('X')
    identity_matrix = np.eye(matrix.shape[0])
    char_poly = sp.det(X *
sp.Matrix(identity_matrix) - sp.Matrix(matrix))
    return X, char_poly

def main():
    try:
        n = int(input("Enter the size of the square
matrix (n): "))
        if n <= 0:
            print("Please enter a positive integer for the
matrix size.")
            return

        my_matrix = create_matrix(n)
        X, char_eq =
characteristic_polynomial(my_matrix)
        print(f"Characteristic equation: {char_eq}")

        eigenvalues = sp.solve(char_eq, X)
        eigenvalues_mod = [abs(e) for e in
eigenvalues]
        print(f"Eigenvalues: {eigenvalues}")

        sum_modulus_eigenvalues =
sum(eigenvalues_mod)
        print(f"Sum of Modulus of Eigenvalues:
{sum_modulus_eigenvalues}")

    except ValueError:
        print("Invalid input. Please enter a valid
positive integer.")

if __name__ == "__main__":
    main()
```

References

- Akram, M., and Akmal, R. (2016). Operations on intuitionistic fuzzy graph structures. *Fuzzy Information and Engineering*, 8(4):389-400. <https://doi.org/10.1016/j.fiae.2017.01.001>
- Askari, J. (2023). A note on the Seidel and Seidel Laplacian matrices. *Boletim da Sociedade Paranaense de Matemática*, 41:1-6. <https://doi.org/10.5269/bspm.51593>
- Askari, J., Das, K.C., and Shang, Y. (2024). On the eigenvalues and energy of the Seidel and Seidel Laplacian matrices of graphs. *Discrete Dynamics in Nature and Society*, 2024, 8390307. <https://doi.org/10.1155/2024/8390307>
- Atanosov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20(1):87-96. [https://doi.org/10.1016/S0165-0114\(86\)80034-3](https://doi.org/10.1016/S0165-0114(86)80034-3)
- Balakrishnan, R. (2004). The energy of a graph. *Linear Algebra and Its Applications*, 387:287-295. <https://doi.org/10.1016/j.laa.2004.02.038>
- Basha, S.S., and Kartheek, E. (2015). Laplacian energy of an intuitionistic fuzzy graph. *Indian Journal of Science and Technology*, 8(33):1-7. <https://doi.org/10.17485/ijst/2015/v8i33/79899>
- Bhat, P.G., and D'Souza, S. (2017). Color signless Laplacian energy of graphs. *AKCE International Journal of Graphs and Combinatorics*, 14(2):142-148. <https://doi.org/10.1016/j.akcej.2017.02.003>
- Bondy, J.A., and Murthy, S.R. (1976). *Graph theory with applications*. North Holland. <https://doi.org/10.1007/978-1-349-03521-2>
- Deepa, G., and Chandrasekaran, V.M. (2018). Lower and upper bound of the Laplacian energy with real and complex roots of an intuitionistic fuzzy directed graph. *International Journal of Applied Systemic Studies*, 8(3). <https://doi.org/10.1504/IJASS.2018.096097>
- Divya, N.G., Saju, M., Roshna, A., and Thomas, K.R. (2024). Smart nanomaterials in energy applications. *Tailored Functional Materials for Clean and Sustainable Development*, 1-18. <https://doi.org/10.1201/9781003394761-22>
- D'Souza, S., Gowtham, H.J., Bhat, P.G., and Girija, K.P. (2022). Color Laplacian energy of some cluster graphs. *TWMS Journal of Applied and Engineering Mathematics*, 12(1):247-259.
- Gutman, I. (1978). The energy of a graph. *Ber. Math. Stat. Sect. Forschungsz. Graz*, 103:1-22.
- Gutman, I., and Zhou, B. (2006). Laplacian energy of graphs. *Linear Algebra and Its Applications*, 414(1):29-37. <https://doi.org/10.1016/j.laa.2005.09.008>
- Harary, F. (1969). *Graph theory*. Addison-Wesley. <https://doi.org/10.21236/AD0705364>
- Harishchandra, S., Ramane, K., Ashoka, B., Patil, P.D. (2023). Some relations between energy and Seidel energy of a graph. *Acta Univ. Sapientiae Informatica*, 15(1):46-59. <https://doi.org/10.2478/ausi-2023-0005>
- Hemalatha, R., and Somasundaram, K. (2022). Sombor index of edge corona product of some classes of graphs. *South East Asian Journal of Mathematics and Mathematical Science*, 18(3):307-316. <https://doi.org/10.56827/SEAJMMS.2022.1803.25>
- Hussein, H.S. (2023). The state of the art of nanomaterials and its applications in energy saving. *Bulletin of the National Research Centre*, 47(1):7. <https://doi.org/10.1186/s42269-023-00984-4>

- Jansirani, M.M., Lakshmanan, K., Kumar, L.S., Sivaranjani, K., Kulandaivelu, R., Santhiya, S., Antony Raj, A.S., and Vinodhini, D. (2024). Exploration of $\text{NaI}g$ -closed and $\text{NaI}g$ -open sets in nano ideal topological spaces with practical applications. *Communications on Applied Nonlinear Analysis*, 31(5):390-397. <https://doi.org/10.52783/cana.v31.1058>
- Karunambigai, M.G., and Parvathi, R. (2006). Intuitionistic fuzzy graphs. *Journal of Computational Intelligence, Theory and Applications*, 20:139-150.
- Kavitha, S., and Jayalalitha, G. (2023). Gaussian graceful labeling. *European Chemical Bulletin*, 12(Special issue 4), 8438-8456.
- Kavitha, S., Jayalalitha, G., and Sivaranjani, K. (2024). Enhanced security in public key cryptography: A novel approach combining Gaussian graceful labeling and NTRU public key cryptosystem. *EAI Endorsed Transactions on Internet of Things*, 10:1-7. <https://doi.org/10.4108/eetiot.4992>
- Kuppusamy, V., Shanmugasundaram, M., Bharathi, P.D., Martin-Barreiro, C., Cabezas, X., Leiva, V., and Castro, C. (2024). Addressing a decision problem through a bipolar Pythagorean fuzzy approach: A novel methodology applied to digital marketing. *Heliyon*, 10(3):e23991. <https://doi.org/10.1016/j.heliyon.2024.e23991>
- Li, X., Shi, Y., and Gutman, I. (2012). *Graph energy*. Springer. <https://doi.org/10.1007/978-1-4614-4220-2>
- Meenakshi, S., and Lavanya, S. (2014). A survey on energy of graphs. *Annals of Pure and Applied Mathematics*, 8(2):183-191.
- Naggor, G.A., and Shajitha, B.S. (2010). Degree, order, and size in intuitionistic fuzzy graph. *International Journal of Algorithm Computing and Mathematics*, 3(1):11-16.
- Narayanan, A., and Mathew, S. (2013). Energy of a fuzzy graph. *Annals of Fuzzy Mathematics and Informatics*, 6(3):455-465.
- Oboudi, M.R. (2016). Energy and Seidel energy of graphs. *Match Communication in Mathematical and Computer Chemistry*, 75:291-303.
- Praba, B., and Chandrasekaran, V.M. (2014). Energy of an intuitionistic fuzzy graph. *Italian Journal of Pure and Applied Mathematics*, 32:431-444.
- Ramane, H.S., Jummannaver, R.B., and Gutman, I. (2017). Seidel Laplacian energy of graphs. *International Journal of Applied Graph Theory*, 1:74-82.
- Ramesh, O., and Basha, S.S. (2020). The implementation of cosine similarity measures in decision-making problems by signless Laplacian energy of an intuitionistic fuzzy graph. *European Journal of Molecular & Clinical Medicine*, 7(10):239-251.
- Reddy, N.R., Khan, M.Z., Basha, S.S., Alahmadi, A., Alahmadi, A.H., and Chowdhary, C.L. (2023). The Laplacian energy of hesitancy fuzzy graphs in decision-making problems. *Computer Systems Science & Engineering*, 44(3):2637-2653. <https://doi.org/10.32604/csse.2023.029255>
- Shigehalli, V.S., and Betageri, S.K. (2015). Color Laplacian energy of graphs. *Journal of Computer and Mathematical Sciences*, 6(9):485-494.
- Sivaranjani, K., Shanmuga Sundaram, O.V., and Akalyadevi, K. (2024). Seidel Laplacian energy of fuzzy graphs. *EAI Endorsed Transactions on Energy Web*, 11:1-6. <https://doi.org/10.4108/ew.5297>
- Umapathy, K., Palanivelu, B., Leiva, V., Bharathi, P. D., and Castro, C. (2023). On fuzzy and crisp solutions of a novel fractional pandemic model. *Fractal and Fractional*, 7:528. <https://doi.org/10.3390/fractalfract7070528>
- Yalcin, N.F. (2022). On Seidel Laplacian matrix and energy of graphs. *Acta Univ. Sapientiae Informatica*, 14(1):104-118. <https://doi.org/10.2478/ausi-2022-0007>
- Zadeh, L.A. (1965). Fuzzy sets. *Information Science*, 8:338-353. [https://doi.org/10.1016/S0019-9958\(65\)90241-X](https://doi.org/10.1016/S0019-9958(65)90241-X)