

NUMERICAL COMPARISON OF THREE LOCAL MULTIVARIATE GAUSSIAN-BASED INTERPOLATION SCHEMES

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Abstract

This study introduces the Local Explicit Radial Basis Function scheme (LE-RBF), alongside an investigation of its effectiveness compared with two other Gaussian RBF-based methods: the Modified Radial Basis Function Shepard's method (MRBFS) and the Local Radial Basis Function Karel's algorithm (KAREL), for interpolation problems. The research focuses on three challenging applications: Franke's function in two and three dimensions, and grayscale image reconstruction. All three schemes, operating in a local interpolation manner, are assessed across multiple criteria, including accuracy, CPU time and storage, and sensitivity to parameters and node sizes. The numerical experiments conducted reveal that while all three schemes yield reasonably good results in most scenarios, the proposed LE-RBF method stands out for its higher accuracy, reduced sensitivity to nodes and shape choices, and lower CPU time and storage requirements. Notably, LE-RBF demonstrates superior performance in various node densities and shape parameter-selecting strategies, especially when combined with specific strategies like the Carlson shape at 142×142 nodes. Its efficiency in processing time further highlights its practicality for complex applications. The study concludes that LE-RBF, with its robust performance and flexibility, presents a promising avenue for future application in diverse scientific and engineering tasks.

Keywords: Local interpolation; Gaussian radial basis function; Multivariable functions; Shape parameter; Image reconstruction

Introduction

Interpolation has played a crucial role in a variety of applications in both science and engineering (Sakhnovich, 1997). Although the concept of interpolation seems not that complicated, it is considered to be one of the most useful and powerful

tools for nowadays practical uses for applied mathematics. Moreover, more ideas have continued to emerge over the past decades. Some recent and successful ones include the modified spatiotemporal kriging interpolation method to re-establish the non-

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uniform spatiotemporal correlation distance (Wu *et al.*, 2023), the interpolation of soil heavy metal pollution using a coke plant as a case study (Zeng *et al.*, 2023), the interpolation of small datasets in the sandstone hydrocarbon reservoirs (Malvić *et al.*, 2019), the determination of the spatial distribution of nitrogen compounds in groundwater (Bronowicka-Mielniczuk *et al.*, 2019), the investigation of interpolation with large datasets using radial basis functions (Michal and Vaclav, 2019), the application of the Modified Shepard's Method (MSM) (Rithison and Kaennakham, 2021), and modified local distance-weighted (MLD) method (see also references therein).

The interpolation problem starts with a set of discrete data $X = \{x_i\}_{i=1}^N, x_i \in \mathbb{R}^m$ where for each x_i there is its corresponding real value $y_i \in \mathbb{R}$. Then, the task is to construct a continuous function $\Phi(x): \mathbb{R}^m \rightarrow \mathbb{R}$ such that;

$$\Phi(x_i) = y_i. \quad (1)$$

When $m = 1$, this problem is traditionally approached by different schemes, it all obeys the theory of interpolation. If $x_0, \dots, x_m \in \mathbb{R}$ are distinct, then for any $y_i = f(x_i) \in \mathbb{R}$, there exists a unique polynomial $\Phi(x): \mathbb{R}^m \rightarrow \mathbb{R}$ of degree $\leq N$ such that the above interpolation conditions are satisfied. With this simple concept, this work focuses on completing the task using the so-called 'radial basis function (RBF)' (Chen *et al.*, 2023).

Functions whose values depend only on the distance between two vectors in the domain and are also seen to have a multivariate property called "Radial Basis Functions (RBFs)", φ . For $x \in \mathbb{R}^m$ and $r \in \mathbb{R}$, the function is of the form $\varphi(\|x_i - x_j\|) = \varphi(r_{ij}) \in \mathbb{R}$. It is equivalent to say that on the distance from a point of a given set $\{x_j\}$, and $\varphi(x - x_j) = \varphi(r_j) \in \mathbb{R}$, where r_j is the Euclidean distance defined in $\mathbb{R}^m$ as follows.

$$r_{ij} = \sqrt{(x_i^{(1)} - x_j^{(1)})^2 + (x_i^{(2)} - x_j^{(2)})^2 + \dots + (x_i^{(m)} - x_j^{(m)})^2} \quad (2)$$

Radial Basis Function (RBF) interpolation stands out for its flexibility and smooth handling of irregularly spaced data which is crucial in real-world applications. It effectively navigates high-dimensional spaces, avoiding the curse of dimensionality, and adapts well to complex functions. This combination of adaptability, ease of implementation, and computational efficiency makes RBF a preferred choice in diverse scientific and engineering contexts.

Over decades, RBFs have been widely applied in many contexts of applications with different versions. One of the most popular forms of Radial

Basis Function Networks (RBFNs) is the Gaussian (GA), (Ouyang *et al.*, 2023), and it is expressed as follows.

$$\varphi(r) = \exp(-(\varepsilon r)^2) \quad (3)$$

where ε is a parameter, very often determined by the user. Early works on this path were mostly done in a global manner meaning all the interpolation points are taken into the calculation process altogether. This leads to some common shortcomings such as the ill-posed problem and computational costs can easily be unaffordable when the problem at hand is large. Several attempts have been made to remedy and one of those strategies recognized as promising is to derive and carry it out in local manners (Wendland, 1995; Beatson *et al.*, 1999; Carr *et al.*, 2001), on which this work focusses. The main objectives, consequently, are to propose a new local RBF-based (Radial Basis Function) interpolation method formulated based on the Gaussian, and to compare its effectiveness with two other schemes under a set of challenging benchmark problems.

Methodology

The Schemes Under Investigation

The three local Gaussian-based interpolation schemes under this investigation are briefly introduced in this section.

The Modified Radial Basis Function Shepard's Method (MRBFS)

This local interpolation method was presented by Damiana Lazzaro and Laura B. Montefusco in (Lazzaro, 2002). They modified the well-known Shepard's method with a radial basis function. Suppose $P \subseteq \Omega \subseteq \mathbb{R}^m$ $P = \{x_i | x_i = (x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(m)}), i = 1, 2, 3, \dots, N\}$ be a set of N distinct points where $Y = \{y_i | i = 1, 2, 3, \dots, N\}$ and $y_i = f(x_i)$. Define $F: \Omega \rightarrow \mathbb{R}$ such that $F(x_i) = f(x_i)$. The interpolation function is defined as.

$$F(x) = \sum_{i=1}^N \bar{w}_i(x) Q_i(x) \quad (4)$$

The weights are expressed as.

$$\bar{w}_i = \frac{w_i(x)}{\sum_{j=1}^N w_j(x)} \quad (5)$$

Where

$$w_j = \begin{cases} \left(\frac{(R_{w_i} - r_j)_+}{R_{w_i} r_j} \right)^2, & \text{if } r_j < R_{w_i} \\ 0, & \text{if } r_j \geq R_{w_i} \end{cases} \quad (6)$$

where R_{w_i} is the radius of the influence domain and r_j is the Euclidian distance norm of known points (x_i) and interpolate point. $Q_i(x)$ is the radial basis function interpolant relative to the subset $X_j = \{x_k, x_k^j \in \Omega, j \in I_k\}$, I_k is the set of indexes of the neighbor x_k in the influence domain, and n being the number of known points influence domain

$$Q(x) = \sum_{k=1}^{n-1} c_k \varphi(\|x - x_k^j\|) \quad (7)$$

where $\varphi(\|x - x_k\|)$ is the radial basis function.

The Karel's Local Radial Basis Function Algorithm (KAREL)

This method was presented by Karel Uhler (2005). The idea is to locate known points which are used to interpolate unknown values in the radial basis function interpolation. This algorithm creates an influence domain by creating the square called 'window' which the centre of the square is the interpolated point. The known points inside the square are used as inputs for the RBF method. The interpolation form is shown below.

$$F(x) = \sum_{k=1}^n c_k \varphi(\|x - x_k\|) \quad (8)$$

where n is the number of known points in the influence domain. To honor the scheme's owner, it is referred to in this work as 'KAREL'.

The Proposed Local Explicit Radial Basis Function Scheme (LE-RBF)

Inspired by the well-documented approach in (Guangming Yao *et al.*, 2015), we propose a locally-derived interpolation technique in this work, characterized by its implicit nature. It starts with letting $\{x_i\}_{i=1}^N$ be a given large set of scattered data points (sampling points), and $\{z_j\}_{j=1}^M$ be a set of evaluation points. Suppose the function values $\{f(x_i)\}_{i=1}^M$ at the interpolation points are given; they try to use these known values to obtain the approximate values of $\{f(z_j)\}_{j=1}^M$. Let φ be a radial basis function. Choose all interpolation points in the local domain Ω_j as the centers of the basis functions. Using RBF interpolation on Ω_j , we have the following linear system

$$\begin{bmatrix} f(x_1^{[j]}) \\ f(x_2^{[j]}) \\ \vdots \\ f(x_n^{[j]}) \end{bmatrix} = \begin{bmatrix} \varphi(\|x_1^{[j]} - x_1^{[j]}\|) & \varphi(\|x_1^{[j]} - x_2^{[j]}\|) & \cdots & \varphi(\|x_1^{[j]} - x_n^{[j]}\|) \\ \varphi(\|x_2^{[j]} - x_1^{[j]}\|) & \varphi(\|x_2^{[j]} - x_2^{[j]}\|) & \cdots & \varphi(\|x_2^{[j]} - x_n^{[j]}\|) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(\|x_n^{[j]} - x_1^{[j]}\|) & \varphi(\|x_n^{[j]} - x_2^{[j]}\|) & \cdots & \varphi(\|x_n^{[j]} - x_n^{[j]}\|) \end{bmatrix} \begin{bmatrix} \alpha_1^{[j]} \\ \alpha_2^{[j]} \\ \vdots \\ \alpha_n^{[j]} \end{bmatrix}$$

which can be written in the matrix form as follows

$$\hat{f}_n = \varphi_{n \times n} \alpha^{[j]} \quad (9)$$

Thus,

$$\alpha^{[j]} = \varphi_{n \times n}^{-1} \hat{f}_n \quad (10)$$

Therefore,

$$\begin{aligned} f(z_j) &= \sum_{k=1}^n \alpha_k^{[j]} \varphi(\|z_j - x_k^{[j]}\|) \\ &= [\varphi(\|z_j - x_1^{[j]}\|), \varphi(\|z_j - x_2^{[j]}\|), \dots, \varphi(\|z_j - x_n^{[j]}\|)] \alpha^{[j]} \\ &= [\varphi(\|z_j - x_1^{[j]}\|), \varphi(\|z_j - x_2^{[j]}\|), \dots, \varphi(\|z_j - x_n^{[j]}\|)] \Phi_{n \times n}^{-1} \hat{f}_n \end{aligned} \quad (11)$$

which provides the approximate values $\{f(z_j)\}_{j=1}^M$ for all the evaluation points $\{z_j\}_{j=1}^M$.

Gaussian Shape Selection Strategies

In this investigation, we concentrate on three promising strategies for selecting the shape parameter, which is integral in constructing the Gaussian Radial Basis Function (RBF). The presence of this shape parameter is recognized as crucial in influencing the quality of outcomes.

Hardy's method

This technique of defining an optimal shape was created by Hardy (1971), defined in a rather simple form as expressed below

$$\varepsilon = (0.815d)^2 \quad (12)$$

where $d = (\sum_{k=1}^N d_k)/N$ is the mean distance from each data known point to its nearest neighbor.

Franke's method

The second choice employed in this work was proposed by Franke's R. in 1979, defined as:

$$\varepsilon = 1.25 \frac{D}{\sqrt{N}} \quad (13)$$

where D is the smallest diameter of the circle containing all data points, and N is the number of known points.

Carlson's method

Carlson Shape Parameter method was created by Carlson and Thomas (1991). They applied a quadratic polynomial approximate method to calculate the shape parameter. Firstly, they compute the maximum and the minimum values of the known points and scale data to the unit cube by setting

$$\bar{x}_i^{(m)} = \frac{(x_k^{(m)} - x_{min}^{(m)})}{(x_{max}^{(m)} - x_{min}^{(m)})} \quad (14)$$

Then, compute the least squares quadratic polynomial approximation to fit the known data points.

In a 2D case, the quadratic polynomial form is defined as

$$\begin{aligned} \tilde{y} = & a_1(\bar{x}^{(1)})^2 + a_2(\bar{x}^{(2)})^2 + a_3\bar{x}^{(1)}\bar{x}^{(2)} \\ & + a_4\bar{x}^{(1)} + a_5\bar{x}^{(2)} + a_6. \end{aligned} \quad (15)$$

In a 3D case, the quadratic polynomial form is defined as

$$\begin{aligned} \tilde{y} = & a_1(\bar{x}^{(1)})^2 + a_2(\bar{x}^{(2)})^2 + a_3(\bar{x}^{(3)})^2 \\ & + a_5\bar{x}^{(1)}\bar{x}^{(2)}\bar{x}^{(3)} + a_6\bar{x}^{(1)}\bar{x}^{(2)} + a_7\bar{x}^{(1)}\bar{x}^{(2)} \\ & + a_8\bar{x}^{(2)}\bar{x}^{(3)} + a_9\bar{x}^{(1)} + a_{10}\bar{x}^{(2)} + a_{11}\bar{x}^{(3)} + a_{12}, \end{aligned} \quad (16)$$

where $a_1, a_2, \dots, a_{12}$ are constants.

After that, calculate the values of V such that

$$V = \sum_{i=1}^N \frac{(\tilde{y}_i - \tilde{y}_i)^2}{N} \quad (17)$$

Then, we can calculate the shape parameter as follows

$$\varepsilon = \frac{1}{(1+120V)} \quad (18)$$

The Numerical Algorithm

This study uses 3 local radial basis function methods applied with 3 shape parameters to 3 test cases. The main algorithm is shown in Figure 1.

Node Distribution Manners

In this study, the datasets are divided into two categories: training sets and a testing set. The training sets vary in size for each experiment and are distributed in two ways, either uniformly or non-uniformly, shown in Figure 2. On the other hand, the testing set consists of nodes that are uniformly distributed.

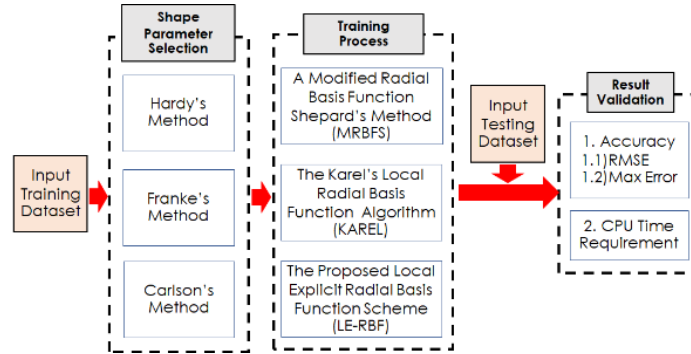


Figure 1. The main algorithm for this study, used for all cases

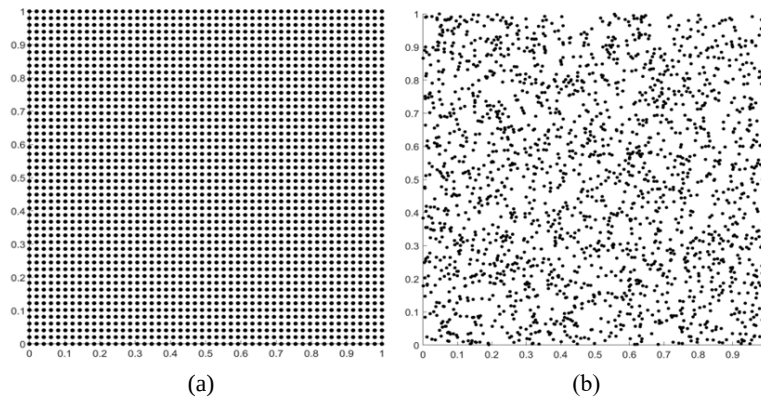


Figure 2. Node distribution styles in two dimensions; (a) Uniformly-distributed and (b) Randomly-distributed

Results and Discussion

This section evaluates the effectiveness of each method by presenting three numerical examples. The performance of these methods is compared and verified against the precise values.

Example 1: The Two-Dimensional Franke's Function

The well-known Franke's function in two dimensions (Franke, 1979) is numerically tackled as the first test case. This function is commonly used to test new methods and is known as a benchmark case. It is mathematically represented as follows

$$f(x, y) = 0.75 \exp \left[-\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4} \right] + 0.75 \exp \left[-\frac{(9x+1)^2}{49} - \frac{(9y+1)^2}{10} \right] + 0.5 \exp \left[-\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4} \right] - 0.2 \exp[-(9x-4)^2 - (9y-7)^2]. \quad (19)$$

The surface plot of this popular function is provided in Figure 3. Table 1 contains the results produced by the three local schemes using two error norms (RMSE and L_∞) for three levels of uniformly distributed nodes (71×71 , 100×100 , and 142×142). As can be seen in this table, both KAREL and LE-RBF methods are clearly found to outperform MRBFS scheme for all levels of node densities and all shape parameter-selecting strategies.

In terms of the requirements of CPU time, on the other hand, it is displayed in Figure 4 that while KAREL (uniform distribution) seems to best produce results quality when more supporting nodes are added, it clearly demands noticeably more time to execute (and effectively, more CPU storage). The proposed scheme, LE-RBF is seen to preserve

reasonably good results with comparatively low amounts of time to process. Figure 4 is promisingly desirable as the less time a scheme requires, the more practical it is for more complex applications. As a comparison amongst the three shape choices, based on overall error magnitudes, Carlson's strategy is noticed to provide better accuracy particularly when more nodes are involved. Taking into consideration both node sizes and shape-selection methods, it is clearly seen that our method used in conjunction with Carlson's shape computed with 142×142 nodes produces the comparatively best results quality with $\text{RMSE} = 8.05\text{E-}6$ and $L_\infty = 5.47\text{E-}05$. Note that to save some space in the article, those results produced in the randomly distributed node cases are not shown here as they show quite similar trends in all aspects to the uniform node distributions.

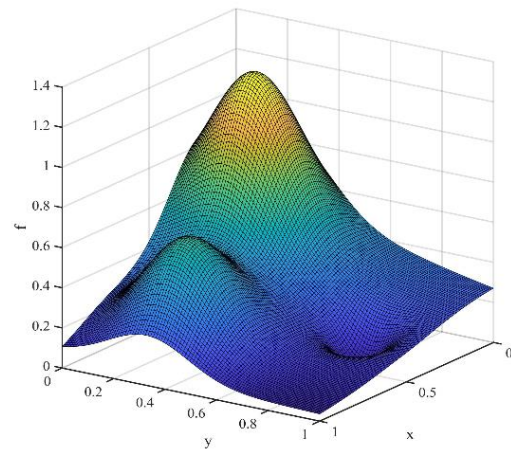


Figure 3. The famous 2D Franke's function surface plot

Table 1. Results quality produced by the three local schemes measured using two error norms (RMSE and L_∞) for three levels of uniformly-distributed nodes (71×71 , 100×100 , and 142×142)

Shape Strategy	Scheme	71×71		100×100		142×142	
		RMSE	L_∞	RMSE	L_∞	RMSE	L_∞
Hardy	MRBFS	7.21E-02	1.08E+00	1.62E-02	7.31E-01	1.59E-02	7.31E-01
	KAREL	2.78E-03	2.65E-02	9.05E-04	3.98E-03	4.41E-04	1.99E-03
	LE-RBF	2.40E-03	1.09E-02	1.21E-03	5.26E-03	6.00E-04	2.58E-03
Franke	MRBFS	7.21E-02	1.08E+00	1.62E-02	7.31E-01	1.59E-02	7.31E-01
	KAREL	1.71E-03	7.96E-03	9.05E-04	3.98E-03	4.38E-04	1.99E-03
	LE-RBF	2.41E-03	1.01E-02	1.21E-03	5.26E-03	6.00E-04	2.58E-03
Carlson	MRBFS	7.21E-02	1.08E+00	1.61E-02	7.31E-01	1.59E-02	7.31E-01
	KAREL	5.63E-05	4.51E-04	2.21E-05	1.14E-04	8.87E-06	6.29E-05
	LE-RBF	1.33E-04	2.43E-03	2.35E-05	1.04E-04	8.05E-06	5.47E-05

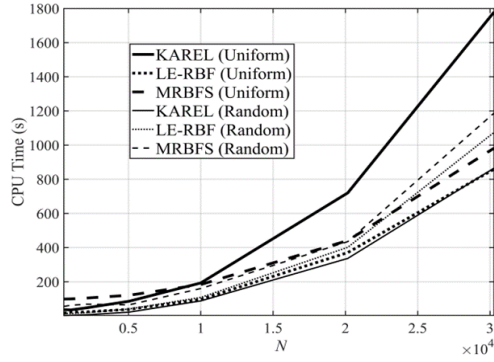


Figure 4. The growth of CPU-time (in seconds) in correspond to the increase in node size; using Hardy shape

Example 2: The Three-Dimensional Franke's Function

Problems involving higher dimensions are known to be more challenging in actual practical uses. For this, the famous Franke's function (Kalani *et al.*, 2023) in three dimensions is taken into our second numerical experiment.

To get more comprehensive insights in terms of the sensitivity to node density, this experiment expands the range of node sizes covering six levels: 23×23 , 32×32 , 45×45 , 71×71 , 100×100 , and 142×142 , for all three shape choices. Using randomly distributed nodes, Table 2 provides both root mean square and maximum errors for all the cases under experiment. As can be seen, our proposed interpolation scheme can maintain comparatively good results even when the number of nodes is changing. This indicates another disable aspect of the scheme which is the node-independent property.

Looking at errors generated by the three shape-selection approaches, all three interpolation methods (MRBFS, KAREL, and LE-RBF) are shown to be slightly affected by the shape choices. LE-RBF, moreover, is seen to lead to the smallest error norm magnitude; $RMSE = 6.57E-04$ and $L_\infty = 9.09E-03$, when deployed in conjunction with the Hardy shape strategy. (Note, once again that the results obtained using uniform nodes behave similarly, so to save some space, they are not shown here.)



Figure 5. Three gray-scale images under investigation; (a) the airport, (b) Lena, and (c) the plane

Example 3: Image Reconstruction Application

To gain more insight in terms of real use of the methods, this last numerical demonstration is engaged with the task of reconstructing damaged Gray-scale images. For this, three popular test images are involved which are the airport, Lena, and the plane, as shown in Figure 5. The original images are corrupted with four levels of damaging (20%, 40%, 50%, and 60%) using the well-known style of image damaging called 'Salt-and-Paper' method (Vimala, 2012). In this manner, the targeted pixels

Table 2. Root mean square error (RMSE) and maximum error (L_∞) measured at different node densities using randomly-distributed nodes for the three-dimensional Franke's function

Node Sizes	Hardy				Franke			Carlson		
	MRBFS	KAREL	LE-RBF	MRBFS	KAREL	LE-RBF	MRBFS	KAREL	LE-RBF	
RMSE										
23x23	5.47E-02	2.75E-02	1.66E-02	4.63E-02	6.29E-02	4.21E-02	4.68E-02	5.86E-02	3.91E-02	
32x32	3.73E-02	2.55E-02	8.75E-03	3.69E-02	3.31E-02	3.01E-02	3.68E-02	3.30E-02	3.01E-02	
45x45	3.14E-02	1.69E-02	6.11E-03	3.10E-02	3.03E-02	2.15E-02	3.06E-02	3.03E-02	2.15E-02	
71x71	2.03E-02	6.64E-03	2.22E-03	2.04E-02	3.01E-02	1.13E-02	1.98E-02	3.01E-02	1.13E-02	
100x100	1.51E-02	6.80E-03	1.31E-03	1.53E-02	2.97E-02	7.51E-03	1.48E-02	2.97E-02	7.51E-03	
142x142	1.23E-02	6.56E-03	6.57E-04	1.23E-02	2.97E-02	4.75E-03	1.20E-02	2.97E-02	4.75E-03	
L_∞										
23x23	6.30E-01	2.13E-01	1.90E-01	3.44E-01	4.93E-01	3.08E-01	3.50E-01	4.73E-01	3.08E-01	
32x32	2.79E-01	7.10E-01	9.80E-02	2.92E-01	3.62E-01	2.17E-01	2.98E-01	3.62E-01	2.17E-01	
45x45	4.06E-01	7.61E-01	6.57E-02	2.49E-01	2.80E-01	1.84E-01	2.52E-01	2.80E-01	1.84E-01	
71x71	2.02E-01	7.65E-02	2.44E-02	1.40E-01	2.69E-01	9.22E-02	1.38E-01	2.69E-01	9.22E-02	
100x100	1.01E-01	8.32E-02	2.03E-02	1.16E-01	2.59E-01	7.36E-02	1.14E-01	2.59E-01	7.36E-02	
142x142	8.81E-02	7.77E-02	9.09E-03	8.33E-02	2.55E-01	4.92E-02	7.84E-02	2.55E-01	4.92E-02	

are converted into either 0 or 255, equation (21), (Kumar *et al.*, 2020).

For some $j = 1, 2, \dots, M$, where M is the number of total pixels, see Figure 6. In actual scenarios, the Salt-and Pepper noise can be resulted from a defect of the camera sensor, software failure, or hardware failure in image capturing or transmission.

Even though three types of images are involved in this experiment, with an overall similar trend of performance,

$$\hat{\Omega}_j(x) = \begin{cases} 0 & \text{'Peppernoise'} \\ 255 & \text{'Saltnoise'} \end{cases} \quad (20)$$

only results obtained for the famous Lena image are displayed and discussed here.

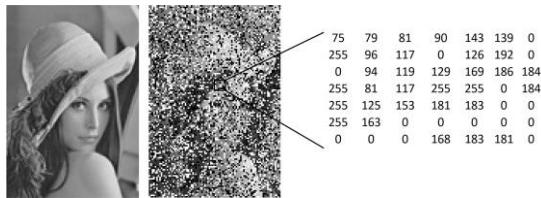


Figure 6. The original 88×128-pixel Lena image (left), with 20%-damage (middle), and the corresponding pixel values (right)

Error measurements (RMSE) for the Lena-image reconstruction application with four levels of damage are provided in Table 3. According to the information contained in this table, all three interpolation schemes are reasonably good when the image is less damaged, and they all lose their ability to reproduce the corrupted pixels when supporting nodes are fewer (i.e., a higher percentage of damage). Being applied in conjunction with those choices of shape selection approaches, the overall trend indicates that the proposed LE-RBF is comparably the best in regenerating the image even at a high percentage of damage. The resulting reconstructed images obtained for each shape type are displayed in Figure 7. With this figure, it is

promising that the proposed interpolation method is a good candidate for implementation to higher dimension and more complex scientific and engineering problems.

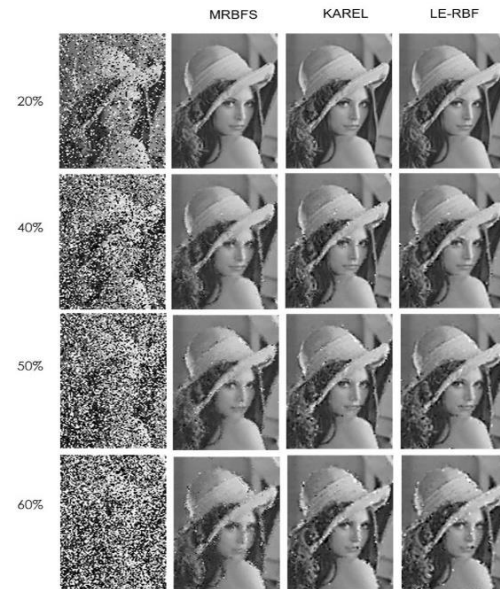


Figure 7. Quality of image regenerated using the three interpolation approaches (MRBFS, KAREL, and LE-RBF) when each image was previously damaged at different levels; 20%, 40%, 50%, and 60%

The research project comparing the performance of three Gaussian-based Interpolation schemes-KAREL, LE-RBF, and MRBFS - using different shape parameter-selecting strategies yields significant insights. The first experiment demonstrates the superior performance of KAREL and LE-RBF over MRBFS across various node densities and shape selection strategies, with KAREL requiring more CPU-time and storage but producing higher quality results, especially with increased node support. LE-RBF stands out for maintaining good results with lower processing time, indicating practicality for complex

Table 3. Error measurements (RMSE) for Lena-image reconstruction application with four levels of damage

% Of damage	Hardy			Franke			Carlson		
	MR BFS	KA REL	LE-RBF	MR BFS	KA REL	LE- RBF	MR BFS	KA REL	LE- RBF
20%	30.53	24.38	21.42	17.54	11.62	11.80	17.27	16.19	13.96
40%	43.33	35.66	30.65	21.85	18.48	14.58	21.30	22.38	19.35
50%	53.68	42.11	38.36	28.29	25.17	15.23	25.43	25.63	22.38
60%	66.56	56.43	54.41	33.69	27.68	18.69	31.39	26.16	31.60

applications. Notably, the Carlson's strategy enhances accuracy, especially with larger node sizes, with LE-RBF and Carlson's shape at 142×142 nodes yielding the best results. Experiment 2 extends the analysis to varying node densities, reinforcing the node-independent quality of the proposed schemes. LE-RBF again leads in minimizing error, particularly with the Hardy's shape strategy. Finally, experiment 3 focuses on Lena-image reconstruction underlines all schemes' effectiveness in less damaged scenarios, with a notable decline in performance as damage increases. Among them, LE-RBF consistently shows superior ability in image regeneration, even at higher damage levels, suggesting its potential for broader applications in complex scientific and engineering problems. These findings collectively underscore the efficacy and practicality of LE-RBF, especially when combined with specific shape selection strategies, in diverse interpolation scenarios.

Conclusion

Interpolation schemes play a crucial role in real-world applications, offering essential tools for data approximation and reconstruction in diverse fields. The study of local Gaussian Radial Basis Function (RBF)-based interpolation methods, specifically KAREL, LE-RBF, and MRBFS, highlights their significant advantages in this context. Gaussian RBF schemes, particularly the LE-RBF, have demonstrated superior performance in terms of accuracy and efficiency across various experimental conditions. Their ability to maintain high-quality results across different node densities and shape parameter-selecting strategies, as evidenced in the research, underscores their flexibility and robustness. LE-RBF uses comparatively lower processing time, coupled with its resilience in handling complex scenarios such as image reconstruction with varying levels of damage, makes it an attractive option for real-world applications. This study's findings not only reinforce the potential of local Gaussian RBF-based schemes in practical applications but also pave the way for their implementation in more complex, multidimensional scientific and engineering problems, thereby broadening the scope of their utility and impact.

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