

BIOOPERATION ON δP_S -OPEN SETS IN TOPOLOGICAL SPACES

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Abstract

The concept of a q operation on a new class of δP_S -open subsets was defined. The comparisons were made with all other existing open sets with this new open set. Here for the same topological space two different operations q and q' are defined and a new concept of bioperation was introduced. The concepts of $[q, q']$ -open sets and $[q, q']$ -regular spaces along with some topological properties were analysed. Two types of $[q, q']$ -closures were studied with their properties, and the submaximal spaces were analyzed.

Keywords: Bioperation; closure; interior; regular space; submaximal space; δP_S -open set; q -open

Introduction

Kasahara (1979) defined the concept of α -closed graphs of a function and explored an operation approach on τ . Ogata (1991) had renamed the operation α to operation γ operation on τ . For the same topological space two different operations γ and γ' are defined and a new concept of bioperation was introduced by Umeshara *et al.* (1992) using union of γ and γ' values. The same was assumed as another type of bioperation using intersection of γ and γ' values by them. Bioperation on α -open sets was analyzed by Khalaf *et al.* (2015). A new class of sets known as δP_S -open sets were introduced by Vidhyapriya *et al.* (2020) by combining the ideas of delta-pre opened P_S -open sets. The concept of a q operation on the collection of δP_S -open sets called q -open sets in topological spaces was proposed by Shanmugapriya *et al.* (2021). Some of the

topological properties of q -open set was defined by Shanmugapriya and Sivakamsundari (2023). Some more properties of q -open set were also analysed by Shanmugapriya and Sivakamsundari. In this article the concepts of $[q, q']$ -open sets and $[q, q']$ -regular spaces and some of its topological properties were analysed. Two types of $[q, q']$ -closures were studied with their properties, and the submaximal spaces were analyzed.

Preliminaries

The definition of δP_S -open set and its properties were discussed in (Vidhyapriya *et al.*, 2020) q -open sets and its properties were discussed in (Shanmugapriya *et al.*, 2021; Shanmugapriya and

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Sivakamasundari, 2023; Shanmugapriya and Sivakamasundari, 2023).

Definition 2.1 Shanmugapriya *et al.* (2021). A δP_S -preopen subset A of a space X is called as a δP_S -open set if for all $x \in A$ there occurs a semiclosed set F in a way that $x \in F \subseteq A$.

The collection of all δP_S -open sets is denoted by $\delta P_S O(X, \tau)$ or $\delta P_S O(X)$.

Definition 2.2 Shanmugapriya *et al.* (2021). An operation ϱ on $\delta P_S O(X)$ is a mapping $\varrho: \delta P_S O(X) \rightarrow \mathcal{P}(X)$ such that $H \subseteq H^\varrho$ for every $H \in \delta P_S O(X)$ where $\mathcal{P}(X)$ is the power set of X and H^ϱ is the value of H under ϱ .

Definition 2.3 Shanmugapriya *et al.* (2021). Let (X, τ) be a topological space and $\varrho: \delta P_S O(X) \rightarrow \mathcal{P}(X)$ be an operation on $\delta P_S O(X)$. A nonempty set A of X is called a ϱ -open set if for all $x \in A$ there occurs a δP_S -open set H such that $x \in H$ and $H^\varrho \subseteq A$.

Proposition 2.4 Shanmugapriya *et al.* (2021). Every ϱ -open set is a δP_S -open set.

Proposition 2.5 Shanmugapriya *et al.* (2021). The union of any class of ϱ -open sets in X is ϱ -open.

Definition 2.6 Shanmugapriya *et al.* (2021). Let (X, τ) be any topological space. An operation ϱ on $\delta P_S O(X)$ is said to be *regular* $_{\varrho}$ -operation if for each $x \in X$ and for every pair of δP_S -open sets H_1 and H_2 in a way that both containing x there occurs a δP_S -open set F containing x such that $F^\varrho \subseteq H_1^\varrho \cap H_2^\varrho$.

Definition 2.7 Shanmugapriya *et al.* (2021). A topological space (X, τ) with an operation ϱ on the collection of $\delta P_S O(X)$ is said to be ϱ -regular space if for given $x \in X$ and for each δP_S -open set H containing x , there occur a δP_S -open set F containing x such that $F^\varrho \subseteq H$.

Definition 2.8 Vidhyapriya *et al.* (2020) The point x in X is called δP_S -closure of A if and only if $A \cap H \neq \emptyset$ for every δP_S -open set H having x . It is denoted by $\delta P_S Cl(A)$.

Statement 2.9 Shanmugapriya *et al.* (2021). If ϱ is *regular* $_{\varrho}$ -operation on the collection of $\delta P_S O(X)$ then $Cl_{\varrho}(A) \cup Cl_{\varrho}(B) = Cl_{\varrho}(A \cup B)$ for subsets A and B of X .

Definition 2.10 Shanmugapriya *et al.* (2021). An operation ϱ on $\delta P_S O(X)$ is said to be an open- ϱ -operation if for every δP_S -open set H having x , there occur a ϱ -open set G containing x such that $G \subseteq H^\varrho$.

Throughout this article ϱ and ϱ' be two operations defined on the collection of δP_S -open set in topological spaces (X, τ) .

$[\varrho, \varrho']$ -open sets

Definition 3.1. A topological space (X, τ) is equipped with two operations namely ϱ and ϱ' defined on a δP_S -open set is called bioperation topological spaces on the collection of δP_S -open set and it is denoted by $(X, \tau, \varrho, \varrho')$.

Definition 3.2. Let ϱ and ϱ' be two operations on the collection of δP_S -open sets of (X, τ) . A non-empty subset A of X is said to be $[\varrho, \varrho']$ -open if for each $x \in A$ there occur a pair of δP_S -open sets U and V having x in a way that $U^\varrho \cap V^{\varrho'} \subseteq A$. The collection of all $[\varrho, \varrho']$ -open sets is denoted by $[\varrho, \varrho'] O(X)$.

Statement 3.3. Let A and B be two subsets of (X, τ) then the subsequent are true:

- a. If A is ϱ -open and B is ϱ' -open, then $A \cap B$ is $[\varrho, \varrho']$ -open.

- b. If A_i is $[\varrho, \varrho']$ -open for every $i \in I$, then $\cup \{A_i | i \in I\}$ is $[\varrho, \varrho']$ -open.
- c. If A is ϱ -open then A is $[\varrho, \varrho']$ -open for any operation ϱ' .
- d. If A is ϱ -open if and only if A is $[\varrho, \mu]$ -open when $\mu: \delta P_S O(X) \rightarrow \mathcal{P}(X)$ is the operation defined by $U^\mu = X$ for every $U \in \delta P_S O(X)$. we denote this operation $[\varrho, \mu]$ as $[\varrho, X]$ also.
- e. If A is $[\varrho, \varrho']$ -open then A is δP_S -open.

Proof. (a) Let A be ϱ -open, then for each $x \in A$, there occurs a δP_S -open set U in a way that $x \in U$ and $U^\varrho \subseteq A$ (1).

Let B be ϱ' -open, then for each $x \in B$, there occurs a δP_S -open set V in a way that $x \in V$ and $V^{\varrho'} \subseteq B$ (2).

Let $x \in A \cap B$, implies $x \in A$ and $x \in B$. Then from (1) and (2) there occurs a pair of δP_S -open sets U and V having x in a way that $U^\varrho \subseteq A$ and $V^{\varrho'} \subseteq B$.

Hence, $U^\varrho \cap V^{\varrho'} \subseteq U^\varrho \subseteq A$ and $U^\varrho \cap V^{\varrho'} \subseteq V^{\varrho'} \subseteq B$.

Therefore, $U^\varrho \cap V^{\varrho'} \subseteq A \cap B$. Thus, $A \cap B$ is $[\varrho, \varrho']$ -open.

(b) Let $x \in \cup A_i \Rightarrow x \in A_i$, for some i and since we have A_i is $[\varrho, \varrho']$ -open then there occurs a pair of δP_S -open sets U and V having x in a way that $U^\varrho \cap V^{\varrho'} \subseteq A_i$ implies

$U^\varrho \cap V^{\varrho'} \subseteq A_i \subseteq \cup A_i$. Hence $\cup A_i$ is $[\varrho, \varrho']$ -open.

(c) Let $x \in A$, since A is ϱ -open, then there occurs a δP_S -open set U in a way that $x \in U$ and $U^\varrho \subseteq A$. Also X is ϱ' -open for any operation ϱ' . Therefore $x \in X$ and there occurs a δP_S -open set X in a way that $X^{\varrho'} \subseteq X$. Thus, $U^\varrho \cap X^{\varrho'} = U^\varrho \subseteq A$.

Hence A is $[\varrho, \varrho']$ -open for any operation ϱ' .

(d) The necessity condition is obvious from (c). On the other hand, let A be $[\varrho, \mu]$ -open when $\mu: \delta P_S O(X) \rightarrow \mathcal{P}(X)$ is the operation defined by $V^\mu = X$ for every $V \in \delta P_S O(X)$. Then for every $x \in A$, there occur δP_S -open sets U and V having x in a way that $U^\varrho \cap V^\mu \subseteq A$, thus $U^\varrho \cap X \subseteq A$ implies $U^\varrho \subseteq A$. Hence A is ϱ -open.

(e) Let A be $[\varrho, \varrho']$ -open and then by Definition 3.2 for each $x \in A$ there occur a pair of δP_S -open sets U and V having x in a way that $U^\varrho \cap V^{\varrho'} \subseteq A$. And we know that $U \subseteq U^\varrho$ and $V \subseteq V^{\varrho'}$ in a way that $U \cap V \subseteq A$ and hence $U \cap V$ is also a δP_S -open set. Hence A is δP_S -open.

Remark 3.4. $[\varrho, X]O(X) = \varrho O(X)$.

Proof. The proof follows from Statement 3.3 (d).

Statement 3.5. Let ϱ and ϱ' be *regular* _{ϱ} -operations.

- a. If A and B are $[\varrho, \varrho']$ -open then $A \cap B$ is $[\varrho, \varrho']$ -open.
- b. $[\varrho, \varrho'] O(X)$ is a topology on X .

Proof. (a) Let A and B be $[\varrho, \varrho']$ -open sets and let $x \in A \cap B$ implies, $x \in A$ and $x \in B$. For every $x \in A$ and since A is an $[\varrho, \varrho']$ -open set there occur a pair of δP_S -open sets U and V having x in a way that $U^\varrho \cap V^{\varrho'} \subseteq A$. And similarly for every $x \in B$ and since B is an $[\varrho, \varrho']$ -open set then there occurs a pair of δP_S -open sets S and T having x in a way that $S^\varrho \cap T^{\varrho'} \subseteq B$.

Therefore, $(U^\varrho \cap V^{\varrho'}) \cap (S^\varrho \cap T^{\varrho'}) \subseteq$

$A \cap B \Rightarrow (U^\varrho \cap S^\varrho) \cap (V^{\varrho'} \cap T^{\varrho'}) \subseteq A \cap B$.

Since ϱ and ϱ' be *regular* _{ϱ} -operations, there occurs δP_S -open sets C and D having x in a way that $C^\varrho \subseteq U^\varrho \cap S^\varrho$ and $D^{\varrho'} \subseteq V^{\varrho'} \cap T^{\varrho'}$. Implies, $C^\varrho \cap D^{\varrho'} \subseteq A \cap B$. Hence, $A \cap B$ is $[\varrho, \varrho']$ -open.

(b) The proof follows from (a) and Statement 3.3(b).

Note: The subsequent example shows that the above statement is not true if ϱ and ϱ' fail to be $regular_{\varrho}$ -operations.

Assume the space $X = \{h, i, j, k\}$ and $\tau = \{\emptyset, X, \{h\}, \{j\}, \{h, i\}, \{h, j\}, \{h, i, j\}, \{h, j, k\}\}$
 $\delta P_S O(X) = \{\emptyset, X, \{i\}, \{j\}, \{h, i\}, \{i, j\}, \{h, i, j\}, \{i, j, k\}\}$. Let ϱ and ϱ' be two operation on the collection of δP_S -open sets of (X, τ) for every $H \in \delta P_S O(X)$

$$H^{\varrho} = H^{\varrho'} = \begin{cases} X, & \text{if } H = \{i\} \text{ and } H = \{i, j\} \\ H, & \text{Otherwise} \end{cases}$$

Here, ϱ and ϱ' are not $regular_{\varrho}$ -operations.

Thus, $[\varrho, \varrho']O(X) = \{\emptyset, X, \{i\}, \{j\}, \{h, i, j\}, \{i, j, k\}\}$ here $\{h, i, j\}$ and $\{i, j, k\}$ are $[\varrho, \varrho']$ -open set but $\{h, i, j\} \cap \{i, j, k\} = \{i, j\} \notin [\varrho, \varrho']O(X)$.

Definition 3.6. A space (X, τ) is said to be $[\varrho, \varrho']$ -**regular** if for each point $x \in X$ and every δP_S -open set H having x there occur a pair of δP_S -open sets U and V having x in a way that $U^{\varrho} \cap V^{\varrho'} \subseteq H$.

Statement 3.7. For a ϱ -regular and ϱ' -regular space and $[\varrho, \varrho']$ -regular space the subsequent properties hold:

- (X, τ) is a $[\varrho, \varrho']$ -regular space if and only if $[\varrho, \varrho']O(X) \supseteq \delta P_S O(X)$.
- (X, τ) is a $[\varrho, \sigma]$ -regular space if and only if (X, τ) is ϱ -regular space.
- If (X, τ) is ϱ -regular and ϱ' -regular then it is $[\varrho, \varrho']$ -regular.
- If (X, τ) is a ϱ -regular space and A is $[\varrho, \varrho']$ -open for an operation ϱ' then A is ϱ -open.

Proof. (a) Let $[\varrho, \varrho']O(X) \supseteq \delta P_S O(X)$ and $x \in X$ and H be a δP_S -open set and $x \in H$, then H is $[\varrho, \varrho']$ -open. By Definition 3.2, there occur a pair of δP_S -open sets U and V having x in a way that $U^{\varrho} \cap V^{\varrho'} \subseteq H$, then by Definition 3.6, (X, τ) is a $[\varrho, \varrho']$ -regular space.

On the other hand, (X, τ) is a $[\varrho, \varrho']$ -regular space and let $x \in X$ and H be a δP_S -open set having x . Then by Definition 3.6 there occur δP_S -open sets U and V having x in a way that $U^{\varrho} \cap V^{\varrho'} \subseteq H$. Hence by Definition 3.2, H is $[\varrho, \varrho']$ -open also. Thus, $[\varrho, \varrho']O(X) \supseteq \delta P_S O(X)$.

(b) Let (X, τ) be a $[\varrho, \sigma]$ -regular space and H be a δP_S -open set having x . Since X is a $[\varrho, \sigma]$ -regular space, there occur a δP_S -open sets U and V in a way that $U^{\varrho} \cap V^{\sigma} \subseteq H$ and $V^{\sigma} = X$. Hence $U^{\varrho} \cap X \subseteq H \Rightarrow U^{\varrho} \subseteq H$. Hence (X, τ) is a ϱ -regular space.

On the other hand, let (X, τ) is ϱ -regular and U be a δP_S -open set having x . Then, there occurs a δP_S -open set F having x in a way that $F^{\varrho} \subseteq H$ by Definition 2.7. Also $H \supseteq F^{\varrho} = F^{\varrho} \cap X = F^{\varrho} \cap X^{\sigma}$ implies, (X, τ) is a $[\varrho, \sigma]$ -regular space.

(c) Let (X, τ) be both ϱ -regular and ϱ' -regular spaces. Let $x \in X$ and H be a δP_S -open set having x then there occurs a δP_S -open set U having x in a way that $U^{\varrho} \subseteq H$ and also there occurs a δP_S -open set V having x in a way that $V^{\varrho'} \subseteq H$. Hence, $U^{\varrho} \cap V^{\varrho'} \subseteq H$, thus (X, τ) is a $[\varrho, \varrho']$ -regular space.

(d) Let (X, τ) be a ϱ -regular space and A is $[\varrho, \varrho']$ -open for an operation ϱ' . Let $x \in X$ and H be a δP_S -open set having x then there occurs a δP_S -open set F having x in a way that $F^{\varrho} \subseteq H$ also, let A be $[\varrho, \varrho']$ -open then by Definition 3.2, for each $x \in A$ there occur a pair of δP_S -open sets U and V having x in a way that $U^{\varrho} \cap V^{\varrho'} \subseteq A \Rightarrow U \subseteq U^{\varrho} \cap V \subseteq V^{\varrho'} \subseteq A \Rightarrow U \cap V = H \subseteq A$, since $F^{\varrho} \subseteq H \subseteq A$. Hence A is ϱ -open.

$[\varrho, \varrho']$ -closures and interiors

Definition 4.1. A subset D of (X, τ) is said to be $[\varrho, \varrho']$ -closed set if its complement $X - D$ is $[\varrho, \varrho']$ -open. The collection of all $[\varrho, \varrho']$ -closed sets is denoted by $[\varrho, \varrho']C(X)$.

Definition 4.2. For a subset A of (X, τ) , $[\varrho, \varrho']Cl(A)$ denotes the intersection of all $[\varrho, \varrho']$ -closed sets of (X, τ) having A . i.e, $[\varrho, \varrho']Cl(A) = \cap \{D | A \subseteq D, D \in [\varrho, \varrho']C(X)\}$.

Statement 4.3. For a point $x \in X$, $x \in [\varrho, \varrho']Cl(A)$ if and only if $C \cap A \neq \emptyset$, for every $[\varrho, \varrho']$ -open set C having x .

Proof. Assume, $x \in [\varrho, \varrho']Cl(A)$ and suppose $C \cap A = \emptyset$ for some $[\varrho, \varrho']$ -open set C of X having x . Then $A \subseteq X - C$ and $X - C$ is $[\varrho, \varrho']$ -closed set in X . Then $[\varrho, \varrho']Cl(A) \subseteq X - C$, by Definition 4.2. Thus, $x \in X - C$, which is a contradiction. Hence $C \cap A \neq \emptyset$ for every $[\varrho, \varrho']$ -open set C having x .

To prove the converse, if $x \notin [\varrho, \varrho']Cl(A)$ then there occurs a $[\varrho, \varrho']$ -closed set D in a way that $A \subseteq D$ and $x \notin A$ so $x \notin D$. Then $X - D$ is a $[\varrho, \varrho']$ -open set in a way that $x \in (X - D)$ and $A \cap (X - D) = \emptyset$ contradicting our hypothesis. Thus $x \in [\varrho, \varrho']Cl(A)$.

Statement 4.4. Let A and B be two subsets of a topological space (X, τ) , then the subsequent properties are true:

- $A \subseteq [\varrho, \varrho']Cl(A)$.
- If $A \subseteq B$, then $[\varrho, \varrho']Cl(A) \subseteq [\varrho, \varrho']Cl(B)$.
- $A \in [\varrho, \varrho']C(X)$ if and only if $[\varrho, \varrho']Cl(A) = A$.

- $[\varrho, \varrho']Cl(A) \in [\varrho, \varrho']C(X)$.

Proof. (a) Obvious.

(b) This follows from Definition 4.2.

(c) $A \in [\varrho, \varrho']C(X)$ if and only if A is the smallest $[\varrho, \varrho']$ -closed set having A . Hence by Definition 4.2, $[\varrho, \varrho']Cl(A) = A$.

(d) Let $A \in [\varrho, \varrho']Cl(A)$ by Definition 4.2 $[\varrho, \varrho']Cl(A) = \cap \{\text{all } [\varrho, \varrho']\text{-closed sets having } A\}$. Hence by (c) $[\varrho, \varrho']Cl(A)$ is a $[\varrho, \varrho']$ -closed set. Therefore, $[\varrho, \varrho']Cl(A) \in [\varrho, \varrho']C(X)$.

Definition 4.5. For a subset A of (X, τ) , $Cl_{[\varrho, \varrho']}(A)$ is defined as $Cl_{[\varrho, \varrho']}(A) = \{x \in X | (U^\varrho \cap V^{\varrho'}) \cap A \neq \emptyset, \text{ for every pair of } \delta P_S\text{-open sets } U \text{ and } V \text{ having } x\}$.

Statement 4.6. In Definitions 4.1, 4.2 and 4.5, put $\varrho' = X$, then for a subset A of X the subsequent properties hold:

- $[\varrho, X]Cl(A) = \varrho Cl(A)$.
- $[\varrho, X]C(X) = \{D | D \text{ is } \varrho\text{-closed}\}$.
- $Cl_{[\varrho, X]}(A) = Cl_\varrho(A)$.

Proof. (a) By Remark 3.4, $[\varrho, X]O(X) = \varrho O(X)$, hence $[\varrho, X]Cl(A) = \varrho Cl(A)$.

(b) By Definition 4.1, $[\varrho, \varrho']C(X)$ is the collection of $[\varrho, \varrho']$ -closed sets. By Remark 3.4, $[\varrho, X]O(X) = \varrho O(X)$, implies $[\varrho, X]$ -closed set is ϱ -closed. Therefore, $[\varrho, X]C(X) = \{D | D \text{ is } \varrho\text{-closed}\}$.

(c) $Cl_{[\varrho, X]}(A) = \{x \in X | (U^\varrho \cap V^X) \cap A \neq \emptyset, \text{ for every pair of } \delta P_S\text{-open sets } U \text{ and } V \text{ having } x\}$
 $= \{x \in X | (U^\varrho \cap X) \cap A \neq \emptyset, \text{ for every } \delta P_S\text{-open set } U \text{ having } x\} = \{x \in X | U^\varrho \cap A \neq \emptyset, \text{ for every } \delta P_S\text{-open set } U \text{ having } x\} = Cl_\varrho(A)$.

Statement 4.7.

- a. For a subset A of (X, τ) , $Cl_{[\varrho, \varrho']}(A) \subseteq [\varrho, \varrho']Cl(A)$ but not conversely.
- b. $\delta P_S Cl(A) \subseteq Cl_{[\varrho, \varrho']}(A)$.

Proof. (a) Let $x \in Cl_{[\varrho, \varrho']}(A)$ and C be a $[\varrho, \varrho']$ -open set having x . By Definition 3.2. for each $x \in C$ there occur a pair of δP_S -open sets U and V having x in a way that $U^\varrho \cap V^{\varrho'} \subseteq C$. By Definition 4.5, $(U^\varrho \cap V^{\varrho'}) \cap A \neq \emptyset \Rightarrow C \cap A \neq \emptyset$. Therefore, by Statement 4.3, $x \in [\varrho, \varrho']Cl(A)$.

(b) Assume $x \in \delta P_S Cl(A)$, then by Definition 2.8, $A \cap H \neq \emptyset$, for every δP_S -open set H having x . Always $H \subseteq H^\varrho$, which implies $A \cap H \neq \emptyset \subseteq A \cap H^\varrho \neq \emptyset$. Similarly for another operation ϱ' we have, $A \cap H^{\varrho'} \neq \emptyset$. Thus, $A \cap (H^\varrho \cap H^{\varrho'}) \neq \emptyset$. By Definition 4.5, $x \in Cl_{[\varrho, \varrho']}(A)$. Thus $\delta P_S Cl(A) \subseteq Cl_{[\varrho, \varrho']}(A)$.

Remark 4.8. $[\varrho, \varrho']Cl(A) \not\subseteq Cl_{[\varrho, \varrho']}(A)$.

Statement 4.9. For a subset A of (X, τ) , then the subsequent Statements are equivalent:

- a. $[\varrho, \varrho']Cl(A) = A$.
- b. $Cl_{[\varrho, \varrho']}(A) = A$.
- c. A is $[\varrho, \varrho']$ -closed, i.e,
 $A \in [\varrho, \varrho']C(X)$.

Proof. (a) $\Rightarrow$ (b) Let $[\varrho, \varrho']Cl(A) = A$.

By Statement 4.7, $Cl_{[\varrho, \varrho']}(A) \subseteq [\varrho, \varrho']Cl(A)$ thus,
 $A \subseteq Cl_{[\varrho, \varrho']}(A) \subseteq [\varrho, \varrho']Cl(A) = A$.

Hence, $Cl_{[\varrho, \varrho']}(A) = A$.

(b) $\Rightarrow$ (c) Let $Cl_{[\varrho, \varrho']}(A) = A$. To prove A is $[\varrho, \varrho']$ -closed, i.e. $X - A$ is $[\varrho, \varrho']$ -open.

Let $x \in X - A \Rightarrow x \notin A = Cl_{[\varrho, \varrho']}(A)$, then there occur δP_S -open sets U and V having x , $(U^\varrho \cap V^{\varrho'}) \cap A = \emptyset \Rightarrow U^\varrho \cap V^{\varrho'} \subseteq X - A$. Then by Definition 3.2, $X - A$ is $[\varrho, \varrho']$ -open. Thus (c).

(c) $\Rightarrow$ (a) Let A is $[\varrho, \varrho']$ -closed, by Definition 4.2, $[\varrho, \varrho']Cl(A) = A$, since A is the smallest $[\varrho, \varrho']$ -closed set having A .

Statement 4.10. For a subset A of (X, τ) , the subsequent statements are true:

- a. If (X, τ) is a $[\varrho, \varrho']$ -regular space, then
 $Cl_{[\varrho, \varrho']}(A) = [\varrho, \varrho']Cl(A) \subseteq \delta P_S Cl(A)$.
- b. $Cl_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A)) =$
 $[\varrho, \varrho']Cl(Cl_{[\varrho, \varrho']}(A)) = [\varrho, \varrho']Cl(A)$.

Proof. (a) By Statement 4.7, $Cl_{[\varrho, \varrho']}(A) \subseteq [\varrho, \varrho']Cl(A)$.

To prove $[\varrho, \varrho']Cl(A) \subseteq Cl_{[\varrho, \varrho']}(A)$ when (X, τ) is a $[\varrho, \varrho']$ -regular space. Let $x \in [\varrho, \varrho']Cl(A)$, then x belongs to every $[\varrho, \varrho']$ -closed set having A . Let D be a $[\varrho, \varrho']$ -closed set in a way that $A \subseteq D$.

Suppose $x \notin D$, then $x \in X - D$, where $X - D$ is $[\varrho, \varrho']$ -open. Since (X, τ) is a $[\varrho, \varrho']$ -regular space then there occur a pair of δP_S -open sets U and V having x in a way that $U^\varrho \cap V^{\varrho'} \subseteq X - D$. Which implies, $(U^\varrho \cap V^{\varrho'}) \cap D = \emptyset$. Thus, $x \notin Cl_{[\varrho, \varrho']}(A)$, which is a contradiction. Therefore $x \in D$. Hence, $Cl_{[\varrho, \varrho']}(A) = [\varrho, \varrho']Cl(A) \subseteq \delta P_S Cl(A)$, by Statement 3.7 (a).

(b) To prove, $[\varrho, \varrho']Cl(Cl_{[\varrho, \varrho']}(A)) = [\varrho, \varrho']Cl(A)$.

We know $A \subseteq Cl_{[\varrho, \varrho']}(A)$, $[\varrho, \varrho']Cl(A) \subseteq [\varrho, \varrho']Cl(Cl_{[\varrho, \varrho']}(A))$ (1)

By Statement 4.7, $Cl_{[\varrho, \varrho']}(A) \subseteq [\varrho, \varrho']Cl(A)$, thus

$[\varrho, \varrho']Cl(Cl_{[\varrho, \varrho']}(A)) \subseteq [\varrho, \varrho']Cl([\varrho, \varrho']Cl(A)) = [\varrho, \varrho']Cl(A)$ (2)

From (1) and (2) $[\varrho, \varrho']Cl(Cl_{[\varrho, \varrho']}(A)) = [\varrho, \varrho']Cl(A)$.

Next to prove, $Cl_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A)) = [\varrho, \varrho']Cl(A) \subseteq Cl_{[\varrho, \varrho']}(A)$. Let $x \in [\varrho, \varrho']Cl(A)$ and let U and V be δP_S -open set having x . Since ϱ and ϱ' are open ϱ -operation there occurs G and G' in a way that, $x \in G \subseteq U^{\varrho}$ and $x \in G' \subseteq V^{\varrho'}$. U^{ϱ} and $V^{\varrho'}$ are ϱ -open and ϱ' -open sets. By Statements 3.3 (a) and 4.3, $U^{\varrho} \cap V^{\varrho'}$ is $[\varrho, \varrho']$ -open set having x , and since $x \in [\varrho, \varrho']Cl(A)$, $(U^{\varrho} \cap V^{\varrho'}) \cap A \neq \emptyset$. Hence $x \in Cl_{[\varrho, \varrho']}(A)$. Therefore (a).

We know that, $A \subseteq \delta P_S Cl(A) \Rightarrow [\varrho, \varrho']Cl(A) \subseteq \delta P_S Cl([\varrho, \varrho']Cl(A)) \subseteq Cl_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A)) \subseteq [\varrho, \varrho']Cl([\varrho, \varrho']Cl(A)) = [\varrho, \varrho']Cl(A)$, by Statement 4.7. Hence $Cl_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A)) = [\varrho, \varrho']Cl(A)$. Therefore, $Cl_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A)) = [\varrho, \varrho']Cl(Cl_{[\varrho, \varrho']}(A)) = [\varrho, \varrho']Cl(A)$.

(b) The proof follows from (a) and 4.10 (b).

Statement 4.11. Let A and B be subsets of X .

- If $A \subseteq B$, then $Cl_{[\varrho, \varrho']}(A) \subseteq Cl_{[\varrho, \varrho']}(B)$.
- $Cl_{[\varrho, \varrho']}(A \cup B) \subseteq Cl_{\varrho}(A) \cup Cl_{\varrho'}(B)$.
- If ϱ and ϱ' are regular ϱ -operation, then $Cl_{[\varrho, \varrho']}(A \cup B) = Cl_{[\varrho, \varrho']}(A) \cup Cl_{[\varrho, \varrho']}(B)$.

Proof. (a) Let $x \in Cl_{[\varrho, \varrho']}(A)$, then for every pair of δP_S -open sets U and V having x in a way that $(U^{\varrho} \cap V^{\varrho'}) \cap A \neq \emptyset \Rightarrow (U^{\varrho} \cap V^{\varrho'}) \cap B \neq \emptyset$. Since $A \subseteq B$. Hence $x \in Cl_{[\varrho, \varrho']}(B)$.

(b) Let $x \in Cl_{[\varrho, \varrho']}(A \cup B)$, then for every pair of δP_S -open sets U and V having x in a way that $(U^{\varrho} \cap V^{\varrho'}) \cap (A \cup B) \neq \emptyset$, then either $U^{\varrho} \cap A \neq \emptyset$ or $V^{\varrho'} \cap B \neq \emptyset$ ----- (1). Otherwise, $U^{\varrho} \cap A = \emptyset$ or $V^{\varrho'} \cap B = \emptyset$, which is a contradiction. Hence by (1), $x \in Cl_{\varrho}(A)$ or $x \in Cl_{\varrho'}(B)$. Thus, $x \in Cl_{\varrho}(A) \cup Cl_{\varrho'}(B)$.

(c). Proof follows from Statement 2.9.

Statement 4.12. Let ϱ and ϱ' be open ϱ -operation and A be any subset of (X, τ) . Then,

- $Cl_{[\varrho, \varrho']}(A) \subseteq [\varrho, \varrho']Cl(A)$.
- $Cl_{[\varrho, \varrho']}(Cl_{[\varrho, \varrho']}(A)) \subseteq Cl_{[\varrho, \varrho']}(A)$.

Proof. (a) By Statement 4.7, we have $Cl_{[\varrho, \varrho']}(A) \subseteq [\varrho, \varrho']Cl(A)$. It is enough to prove that

Definition 4.13. For a subset A of (X, τ) and ϱ and ϱ' be two operations on the collection of $\delta P_S O(X)$. $Int_{[\varrho, \varrho']}(A) = \{x | U^{\varrho} \cap V^{\varrho'} \subseteq A \text{ for some } \delta P_S\text{-open set } U \text{ and } V \text{ having } x\}$.

Statement 4.14. For every subset A of (X, τ) , $Cl_{[\varrho, \varrho']}(X - A) = X - Int_{[\varrho, \varrho']}(A)$.

Proof.

$Cl_{[\varrho, \varrho']}(X - A) = \{x \in X | (U^{\varrho} \cap V^{\varrho'}) \cap (X - A) \neq \emptyset, \text{ for every pair of } \delta P_S\text{-open sets } U \text{ and } V \text{ having } x\} = \{x \in X | (U^{\varrho} \cap V^{\varrho'}) \not\subseteq A, \text{ for every pair of } \delta P_S\text{-open sets } U \text{ and } V \text{ having } x\} = \{x \in X | x \notin Int_{[\varrho, \varrho']}(A)\} = X - Int_{[\varrho, \varrho']}(A)$.

Definition 4.15. A subset N of a topological space (X, τ) is called a $[\varrho, \varrho']$ -neighbourhood of a point $x \in X$, if there occurs a $[\varrho, \varrho']$ -open set C in X in a way that $x \in C \subseteq N$. The collection of all $[\varrho, \varrho']$ -neighbourhood is denoted by $[\varrho, \varrho']N(x)$.

$[\varrho, \varrho']$ -submaximal Spaces

Definition 5.1. A subset A of X is called $[\varrho, \varrho']$ -dense in (X, τ) if $X = [\varrho, \varrho']Cl(A)$.

Definition 5.2. A bioperation-topological space $(X, \tau, \varrho, \varrho')$ is said to be $[\varrho, \varrho']$ -**submaximal space** if every $[\varrho, \varrho']$ -dense subset of X is $[\varrho, \varrho']$ -open.

Example 5.3. Let $X = \{h, i, j\}$ and $\tau = \{\emptyset, X, \{h\}, \{i\}, \{h, i\}\} = \delta P_S O(X)$. Let ϱ and ϱ' be two operation on the collection on δP_S -open sets (X, τ) for every $H \in \delta P_S O(X)$.

$$H^\varrho = \begin{cases} sCl H, & \text{if } h \notin H \\ H, & \text{if } h \in H \end{cases} \quad \text{and} \quad H^{\varrho'} = \begin{cases} gsCl H, & \text{if } H \neq \{h\} \\ H, & \text{if } H = \{h\} \end{cases}$$

Where, $sC(X) = \{\emptyset, X, \{h\}, \{i\}, \{j\}, \{h, j\}, \{i, j\}\} = gsC(X)$

Then, $[\varrho, \varrho']O(X) = \{\emptyset, X, \{h\}, \{i\}, \{h, i\}\}$.

Here $[\varrho, \varrho']$ -dense subset of X are X and $\{h, i\}$ and they are $[\varrho, \varrho']$ -open sets. Hence $(X, \tau, \varrho, \varrho')$ is a submaximal space.

Definition 5.4. A bioperation-topological space $(X, \tau, \varrho, \varrho')$ is said to be $[\varrho, \varrho']$ -**extremely disconnected** if $[\varrho, \varrho']Cl(A) \in [\varrho, \varrho']O(X)$ for every $A \in [\varrho, \varrho']O(X)$.

Statement 5.5. For a bioperation topological space $(X, \tau, \varrho, \varrho')$, the subsequent are equivalent:

- X is $[\varrho, \varrho']$ -extremely disconnected.
- $[\varrho, \varrho']Cl(A) \cap [\varrho, \varrho']Cl(B) = \emptyset$ for every disjoint $[\varrho, \varrho']$ -open sets A and B .
- $[\varrho, \varrho']Cl(A) \cap [\varrho, \varrho']Cl(B) \subseteq [\varrho, \varrho']Cl(A \cap B)$, for every disjoint $[\varrho, \varrho']$ -open sets A and B .
- $[\varrho, \varrho']Cl(Int_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A))) \cap [\varrho, \varrho']Cl(B) = \emptyset$, for every disjoint $[\varrho, \varrho']$ -open sets A and B .

Proof. $(b) \Rightarrow (a)$ Let $[\varrho, \varrho']Cl(A) \cap [\varrho, \varrho']Cl(B) = \emptyset$, for every disjoint $[\varrho, \varrho']$ -open sets A and B . Let C be $[\varrho, \varrho']$ -open set in a way that C and $X - [\varrho, \varrho']Cl(C)$ are disjoint $[\varrho, \varrho']$ -open sets. $[\varrho, \varrho']Cl(C) \cap [\varrho, \varrho']Cl(X - [\varrho, \varrho']Cl(C)) = \emptyset$.

This implies that $[\varrho, \varrho']Cl(U) \subseteq (Int_{[\varrho, \varrho']}([\varrho, \varrho']Cl(U)))$. Thus, $[\varrho, \varrho']Cl(U)$ is $[\varrho, \varrho']$ -open, hence X is $[\varrho, \varrho']$ -extremely disconnected.

$(a) \Rightarrow (c)$ Let C be $[\varrho, \varrho']$ -open set in a way that $[\varrho, \varrho']Cl(U)$ is $[\varrho, \varrho']$ -open set. Let for every disjoint $[\varrho, \varrho']$ -open sets A and B , $[\varrho, \varrho']Cl(A) \cap [\varrho, \varrho']Cl(B) \subseteq [\varrho, \varrho']Cl([\varrho, \varrho']Cl(C) \cap [\varrho, \varrho']Cl(B)) \subseteq [\varrho, \varrho']Cl(A \cap B)$

$(c) \Rightarrow (b)$ Let A and B be disjoint $[\varrho, \varrho']$ -open sets by (c) $[\varrho, \varrho']Cl(A) \cap [\varrho, \varrho']Cl(B) \subseteq [\varrho, \varrho']Cl(A \cap B) = [\varrho, \varrho']Cl(\emptyset) = \emptyset$. Hence $[\varrho, \varrho']Cl(A) \cap [\varrho, \varrho']Cl(B) = \emptyset$.

$(b) \Rightarrow (d)$ Let A and B be disjoint $[\varrho, \varrho']$ -open sets. We know that $Int_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A))$ is $[\varrho, \varrho']$ -open also $Int_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A) \cap B) = \emptyset$, by (b) $[\varrho, \varrho']Cl(Int_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A))) \cap [\varrho, \varrho']Cl(B) = \emptyset$

$(d) \Rightarrow (b)$ Let A and B be disjoint $[\varrho, \varrho']$ -open sets, by (d) $[\varrho, \varrho']Cl(Int_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A))) \cap [\varrho, \varrho']Cl(B) = \emptyset$ we know that, $[\varrho, \varrho']Cl(A) \subseteq [\varrho, \varrho']Cl(Int_{[\varrho, \varrho']}([\varrho, \varrho']Cl(A)))$. Hence $[\varrho, \varrho']Cl(A) \cap [\varrho, \varrho']Cl(B) = \emptyset$.

Conclusion

The bioperation ϱ and ϱ' defined on the collection of δP_S -open sets were defined here. The new open set called $[\varrho, \varrho']$ -open set were discussed along with its properties such as regular spaces, regular operation, closure and interior. Also the submaximal space of this new open set were analysed. In future

the continuity of this bioperation may be define and extended.

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