

LOCAL ALGEBRAIC MULTIGRID TECHNIQUE TO ACCELERATE THE CONVERGENCE OF NUMERICAL COMPUTATION

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Abstract

The delayed convergence of numerical computation on large sparse system of equations is caused by the low-frequency modes error and the numerical methods that deal with entire nodes on domain. This paper proposes a new method of local algebraic multigrid (LAMG) which is developed based on algebraic multigrid (AMG) and adaptive region techniques. The AMG technique used to eliminate the low-frequency modes error while the adaptive region technique used to optimize the computational regions. The deflection which is combined with residual and percentage of the error was used to define the adaptive computation regions. The computational efficiency of the proposed LAMG method was evaluated by investigating on four cases of linear and nonlinear one-dimensional boundary layer equations. The predicted results are then compared with global algebraic multigrid (GAMG) method, the classical AMG, with non-adaptive region treatment. It is found that both methods, proposed LAMG and GAMG, are converged with the same number of computational cycles and same order of accuracy. The proposed LAMG, however, was executed on fewer nodes than GAMG about 13-50% and 6-14% for 1D and 2D problems, respectively, that can decrease in computational time for all the test cases.

Keywords: Algebraic Multigrid, Acceleration Technique, Adaptive Domain, AMG, Local Algebraic Multigrid

Introduction

Multigrid (MG) was the grid strategy designed to promote rapid convergence by using coarser grid systems to cooperate in numerical methods (Fedorenko, 1962; Poussin, 1968; Settari and Aziz 1973). This method can save the computational work by using coarse grids that has effectively eliminate the error in low-frequency modes, which were the major cause of converging delays (Briggs *et al.*, 2000; Shome, 2006). For unstructured grid, however, it is difficult to prepare the multi-level grids and to

interconnect between the grid sets due to irregular grid pattern. The technique of algebraic multigrid (AMG) has been developed for solving those problems by treating on data structure rather than grid structure (Trottenberg *et al.*, 2001; Falgout, 2006; Gilbert, 2009; Feng and Zhao, 2020; Li *et al.*, 2020; Feng and Zhao, 2021; Tan *et al.*, 2021). Due to its flexibility in using the AMG has usually been found as the standard feature including in general commercial software such as FLUENT, CFX, STAR CCM+, etc. They are

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available in various forms such as V-cycle, W-cycle, F-cycle, and also MLAT (Multi-Level adaptive technique) which is the adaptive technique according to Brandt (1977).

Main disadvantage of using classical MG technique is all their grid sets are constructed cover entire domain like global refinement, leading to global relaxation. In general, the fine grid is usually required for improving the computation accuracy. For the region where error has low volatility, however, using of coarse grid may enough to give sufficiently accurate solutions. That is mean that the global refinement is actually may in fact not be necessary. Therefore, the local refinement technique such as AMR (adaptive mesh refinement) has been developed (Brown and Lowe, 2005). The computation began on coarse grid and then finer grids would be built cover the particular regions overlapping the original grid where the correctness of the result is not adequate. Thus, the grid sets on the computational domain are consisted of various grid sizes, so-called composite grid method. Phongthanapanich and Dechaumphai (2008, 2012) has investigated the AMR technique on the hyperbolic and flow problems effectively. Although it is the local refinement technique but the computational processes are still globally treatment. Due to filling of finer grid lay on the coarse grid layers, furthermore, so the behavior of mode error still gives the low frequency that is resulting the delay of convergence.

The combining of multigrid method and local grid refinement technique is the so-called FAC (Fast Adaptive Composite) method. With FAC technique the coarse grid, base grid, has constructed entire domain while high error contamination in local areas covered by the local fine grids. The relaxation has occurred only on the regions where the fine grids are contained. Therefore, FAC method is claimed to be fully local technique that has both local refinement and local relaxation (McCormick, 1984; McCormick and Thomas, 1986).

This paper presents a new numerical method that has highly effective in driving the computation convergence. The method has been developed based on local grid refinement and adaptive region techniques. In each cycle the computed result on global domain has checked. Areas with high error are identified and the new grids are created to support AMG computation. With this proposed technique the AMG has been treated as local treatment so-called local AMG (LAMG). During the computational cycle the regions where the error has lower enough will be waved from the computation. This proposed method is fully local operation that can accelerate the convergence and reduces the computational time. At the first time here, the proposed method will be

investigated on the boundary layer problems to assess the computation potential. The programming code has been developed based on C++ and executed on a computer; Intel(R) Xeon(R) CPU W3550 @3.07GHZ, 8.0 Gb and 64-bit operating system.

Classical Algebraic Multigrid (AMG) Technique

In order to develop the local computation technique, principle of numerical method by using multi-level grid technique must be cleared. Because this technique plays an important role in using for solving the linear equation system,

$$A\phi = b. \quad (1)$$

where A is a real $n \times n$ matrix, ϕ and b are vectors in domain region R^n . There are two main concepts of multigrid technique: (a) the smoothing, which is used to relax the data such as Gauss-Seidel method, and (b) the inter-grid transferring, which is used to transfer the data between grid sets. The well-known approach of the inter-grid transferring is Galerkin operator which can be seen with higher order in Hooshyar (2014). The restriction operator (I_l^{l+1}) uses to transfer the data from fine grid to coarse grid while prolongation operator (I_{l+1}^l) transfers from coarse grid to fine grid. The grid levels $l=1, \dots, L-1$, refer to finest grid to coarsest grid respectively. Technique of AMG quite flexible for constructing the coarse grid particularly unstructured meshing. The steps of solving the linear Equation (1) with AMG by using two-grid correction scheme (Moukalled *et al.*, 2016) are demonstrated as follows:

1. Initial guess for ϕ^l and $l = 1$
2. Smooth on $A^l \phi^l = b^l$
3. Compute residual $r^l = b^l - A^l \phi^l$
4. Solve $A^{l+1} \phi^{l+1} = I_{l+1}^{l+1} r^l$
5. Correct $\phi^l = \phi^l + I_{l+1}^l \phi^{l+1}$
6. Smooth on $A^l \phi^l = b^l$ and return Step-2.

Step-2 performed iteratively and the computed result from this step is used to define the residual error in Step-3, and then used to transfer the data to coarser grid by using restriction operator. Hence, the equation of Step-4 is normally called residual equation. The computations are performed on coarse grid to approximate the solution ϕ^{l+1} , which is used to correct the solution on next finer grid level in Step-5. To construct the interpolation operator on grid level l , fine grid nodes have been selected to be coarse grid on the next grid level by using strong couple conditions (Brezina *et al.*, 2005; Treister,

2014; Brezina *et al.*, 2012; Kalchev, 2012). As C and F are the grid set that contained coarse grid and fine grid, respectively, and $C \cap F = \emptyset$. Indexes i and j are nodal indexes on domain. The inter-grid transferring operator was defined as follows:

$$\phi^l = (I_{l+1}^l \phi^{l+1})_i = \begin{cases} \phi^{l+1} & \text{if } i \in C \\ \sum_{j \in C_i} \omega_{ij} \phi_j^{l+1} & \text{if } i \in F \end{cases} \quad (2)$$

For one-dimensional problem, the weighting factors are defined as $\omega_{ij} = a_{ij} / a_{ii}$ and $A = (a_{ij})_{i,j=1,\dots,n}$.

Local Algebraic Multigrid (LAMG) Technique

In order to improve the AMG method based on local treatment, the domain must be defined into two parts; *active zone* and *mature zone*. Active zone (ω) is the region where the *defection* larger than convergence criteria while the mature zone is the region where the defection smaller than convergence criteria. Determine global domain (Ω), active zone is therefore the region where $\omega \subseteq \Omega$. The nodal defection δ_i is defined as the maximum values between absolute residual and percentage of error as follows:

$$\delta_i = \max(|r_i|, e_i), \quad (3a)$$

where

$$|r_i| = |b - A\phi|_i, \quad (3b)$$

$$e_i = \frac{|\phi_i^{(m)} - \phi_i^{(m-1)}|}{\phi_i^{(m)}} \times 100 \quad (3c)$$

and m defined as cycle number. Therefore, there was also be condition for temporary boundary δ_{BC} . The active zone was the region where defection was larger than temporary boundary condition as follows:

$$\omega = \{\delta_i > \delta_{BC}\} \quad (4)$$

For LAMG technique, the active zone has been specified as single region executed by using multigrid method with temporary boundary conditions as shown in Figure 1. The AMG method, which is the global relaxation technique, has been developed to be the LAMG using local computing

technique. So, there will be the algorithm for determining the region. Two-grid version of LAMG calculation steps are described as follow:

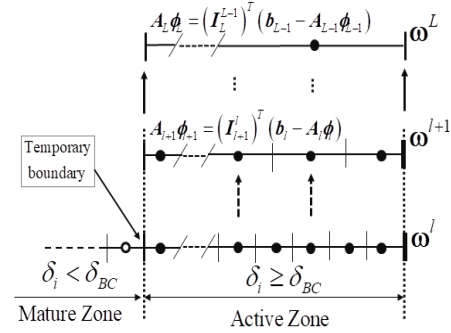


Figure 1. Schematic of the LAMG operation

1. Smooth $A^l \phi^l = b^l$ on global domain Ω
2. Compute residual $r^l = b^l - A^l \phi^l$
3. Define active zone $\omega^l = \{\delta_i > \delta_{BC}\}$
4. Solve $A^{l+1} \phi^{l+1} = I_{l+1}^{l+1} r^l$ on local domain ω^{l+1}
5. Correct $\phi^l = \phi^l + I_{l+1}^l \phi^{l+1}$
6. Smooth $A^l \phi^l = b^l$ on entire domain and return to Step-2

The computation started on finest grid. The defection on the grid, for each element, is evaluated for convergence. Elements with a higher defect than a given condition are defined as the active zone. The remainder will be identified as a mature zone. The boundary between the mature zone and the active zone is defined as the temporary boundary of local domain (Step-4). The domain division in Step-3 was enhanced by the AMG method, making the LAMG method a local computation on a coarse-grid level and the correction in the Step-5 takes place on that particular active zone. On the one hand, if the calculations are in converging behavior, the active zone will be gradually reduced to the end. On the other hand, the mature zone will gradually increase the size and cover the entire domain. The proposed numerical technique incorporates the AMG method with the region improvement technique in order to minimize the computational work and accelerate convergence. The computations take place only for the region needed. Therefore, the technique presented is fully localized operations.

The Evaluations and Test Cases

The LAMG technique has been implemented on the boundary layer equations of Berger (1982) with the general form as follows:

$$\varepsilon \frac{d^2 \phi}{dx^2} + a(x) \frac{d\phi}{dx} + b(x)\phi = f(x); (x_1 < x < x_2). \quad (5)$$

The mode constant $0 < \varepsilon \ll 1$ and $f \geq 0$. The parameters $a(x)$ and $b(x)$ are the continuous functions, and $\phi(x_1) = \phi_1$ and $\phi(x_2) = \phi_2$ are the boundary conditions. This problem is appropriate for evaluating the local improvement technique because the behavior of the solution has suddenly change in some regions. The equations are expressed as cases A, B, C, and D are shown in Equation. (6)-(9), respectively. Case A, B and C are linear problems while case D is non-linear problem.

Case A:

$$\varepsilon \frac{d^2 \phi}{dx^2} - \frac{d\phi}{dx} = 0 \quad (6)$$

$\phi(-1) = 1, \phi(1) = 2, \varepsilon = 0.005.$

Case B:

$$\varepsilon \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} = 0 \quad (7)$$

$\phi(-1) = 1, \phi(1) = 2, \varepsilon = 0.0001.$

Case C:

$$\varepsilon \frac{d^2 \phi}{dx^2} + |x| \frac{d\phi}{dx} + \left(x - \frac{1}{2}\right)^3 \phi = 0 \quad (8)$$

$\phi(-1) = 1, \phi(1) = 2, \varepsilon = 0.01.$

Case D:

$$\varepsilon \frac{d^2 \phi}{dx^2} + \phi \frac{d\phi}{dx} + 0.5|\phi| = 0 \quad (9)$$

$\phi(-1) = 1, \phi(1) = 1.$

In order to evaluate the accuracy and efficiency of the proposed LAMG technique, the Poisson's equation with shape changing of solution on four steep gradient cones in a square domain have been investigated. It was utilized to evaluate the performance of nodeless variable finite element with adaptive meshing technique (Traivivatana *et al.*, 2008). The governing equation is expressed as follow:

$$\nabla^2 \phi = -Q(x, y) \quad (10)$$

where $-0.5 \leq x \leq 0.5$ and $-0.5 \leq y \leq 0.5$ with condition $\phi = 0$ entire domain boundaries. The exact solution of the problem is expressed in Equation (11) and the contour of the solution is shown in Figure 2.

$$\phi(x, y) = \begin{cases} 0 & r > 0.075 \\ 10[1 + \cos(\pi r / 0.075)] & r \leq 0.075 \end{cases} \quad (11)$$

In numerical processes, the first-order derivative terms has been estimated with upwind differencing scheme while the central differencing scheme has been applied on the second-order derivative terms. The computational efficiency of the proposed LAMG method has been assessed against the classical AMG method. The AMG is the non- adaptive region technique. So it is the so-called global algebraic multigrid (GAMG) method. The initial condition is set to be zero and the computing had stops when the defection of any element in the domain had lower than the setting value. In the paper here, the computation will stop as the defection δ less than 10^{-7} (stopping condition). For the proposed LAMG method, the temporary boundary condition of the active zone needs to be small in order to ensure that the computation would be stable. In the paper here the condition at boundaries are assigned equal to the stopping condition, $\delta_{BC} = \delta$.

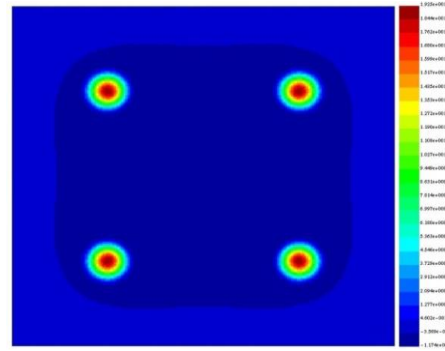


Figure 2. The exact solution contour for the Poisson's equation

Results and Discussions

The LAMG method based on the assumption that convergence was localized behavior. The areas with the high convergence rate, the areas with the low gradient of the closed nodes solution. On the other hand, the higher gradient means that the problem was complex and that the convergence was difficult. The converged area (fewer complexes) therefore is not relaxed in order to save computational work, which is the mature zone. The relaxation is limited to area where the defection is higher than the criteria, that is the active zone. The results of the test cases, for the LAMG method, are shown as the adaptation of the active zone. The active zone is defined by the area covered by the blue background is calculated using multigrid method. The remaining regions are omitted. However, global relaxation for

a few iterations on finest grid is needed to define the new active zone that would be changed for every cycle. The improvement in the active zone of the test cases on 1000 nodes is illustrated as follows.

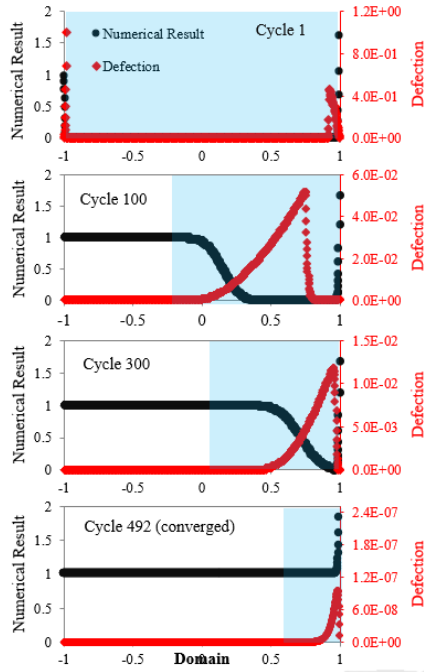


Figure 3. Convergence behaviors of Case A

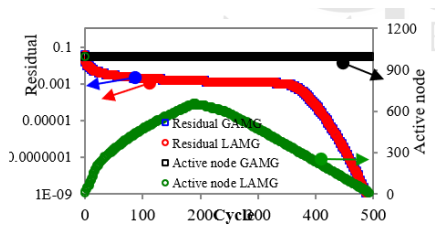


Figure 4. Active nodes relative to residual declination on Case A

For Case A, it is found that the left region was smooth and the right region adjacent to the right boundary was suddenly changed. During the computation cycle, the changing of the active zone illustrated with highlight background and the rest region was the mature zone that is related to the reduction of defection (red symbol). The results of residual reduction during relaxation cycle by LAMG are shown in Figure 3. It is found that the active zone was improved by decreasing the size from left to right. This relates to the reduction of the defection. Due to the left region of the domain has smooth behavior than the right region leading to the convergent more readily than the right one. Figure 4 shows the number of nodes used of each

calculation cycle with respect to the maximum defection. The data refers to their axis are pointed by the arrows. The GAMG has been calculated on fixed grid for each cycle and depicted by black symbol while the AMG by green symbol. It can be seen that the change of residual and number of active nodes are related to the improvement of the active zone size. The area under the graph represented the total number of nodes used until the computation was converged. It can be seen that the LAMG used a much fewer number of nodes than the GAMG, whereas the residual reduction of both methods was decreased at the same amount. It demonstrated that the LAMG took fewer computations, but provided the results comparable to the GAMG method.

In order to determine the accuracy of the results on case A, the computed results obtained from LAMG and GAMG methods have been compared to exact solution as follows:

$$\phi(x) = (1 - \exp(-2/\varepsilon)) + \exp((x-1)/\varepsilon) \quad (12)$$

Figure 5 show the comparison of the absolute error between the computed result and the exact solution. It can be seen that the results of both LAMG and GAMG were consistent and overlaid on the exact solution. There was too small error in the left region and slightly larger error in the right region due to the large gradient variation. Due to the number of nodes used in the region is limited, so it is difficult to detect this sudden change behaviour, resulting to the high error in the right region (Figure 5). The error obtained from LAMG and GAMG, however, were consistent and the maximum with less than 3%. It has also been shown that the LAMG provided the comparable accurate result to the GAMG. For the computational time, the LAMG can save up to 32% compared to GAMG, as shown in Table 1.

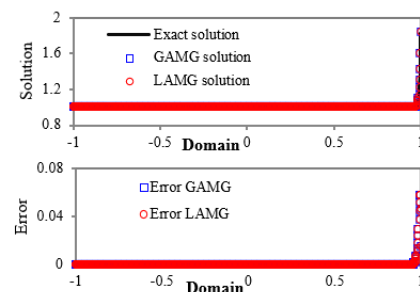


Figure 5. Evaluations of exactness in case A, (Top) numerical results against the exact solution (Bottom) the error of the numerical results

The characteristic of improving the active zone was that it moved while reducing the area. The movement of the active zone can be seen as related to the reduction of the deflection and the last high-deflection field was the zone where the problem was complicated. The reduction of the active zone size reflected an economical calculation. In Figure 6, active zone (highlight region) decreased both sides to the central domain which is the area where the problem was suddenly changed. While the manner on left and right zones were steady. Figure 7 shown the active nodes were used along with the eliminating deflection. The data refers to their axis are pointed by the arrows. It was found that the LAMG used a fewer node (green

symbol) but could maintained the convergence rate compatible to the GAMG method (black symbol).

In case C, Figure 8, it can be seen that the deflection (red symbol) was high where the region which the computed result was not steady. While on the right zone, the result was smooth and the deflection was insufficient. On that region, smooth result, was the first zone that has been converged. Then, the deflection was decreased from right to left domain. The adaptations of the active zone were correlated to the eliminating of deflection. In Figure 9, it can be seen that the declension of deflection for both methods were at the same rates. While the LAMG method reduced the active nodes up to 40% for save computational work.

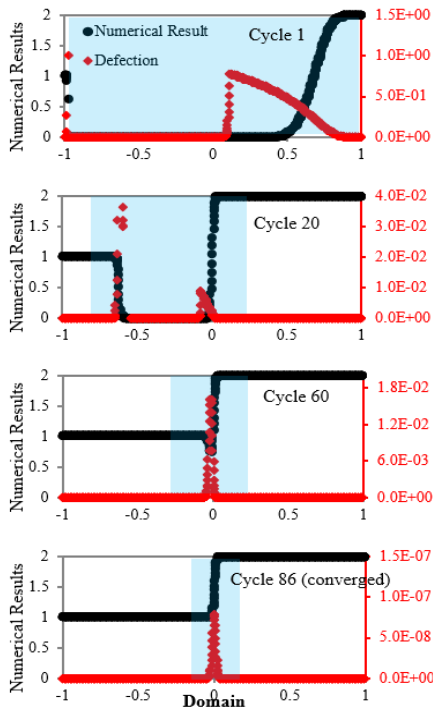


Figure 6. Convergence behaviors of Case B

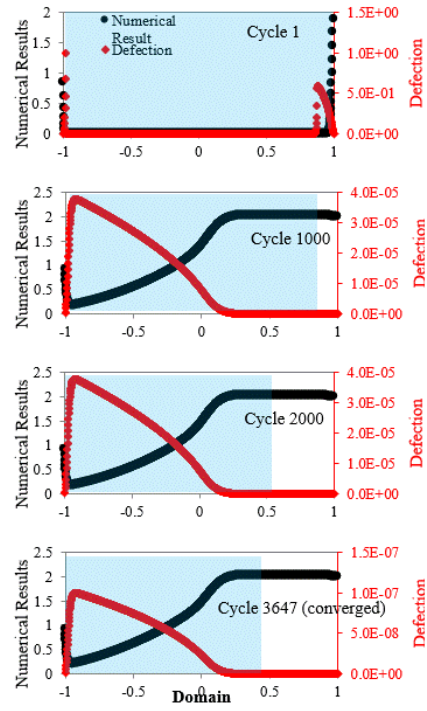


Figure 8. Convergence behaviors of Case C

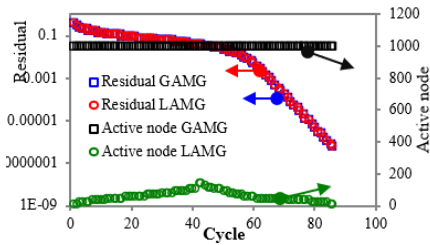


Figure 7. Active nodes relative to residual declension on Case B

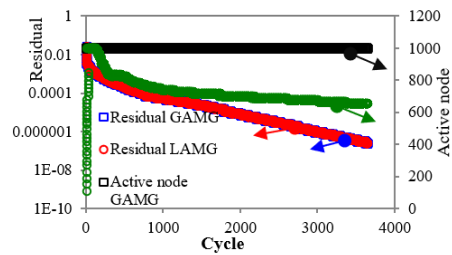


Figure 9. Active nodes relative to residual declension on Case C

In case D, as the equation form, it was a non-linear problem. As Figure 10, active zone can detect the complexity behavior very well. It can be seen that the active zone was decreasing from right to left domain which was the region where smoothest to highest developed. The left domain hence was last region and difficult to converge. In Figure 11, it is found that the LAMG method has been reduced and saved active nodes up to 90% which meant that the saving of computational work and computational time. While the decreasing of the deflection for both methods were still correlated. That shown stability and flexibility of the LAMG method to solve wide variety of problems both linear and non-linear very well.

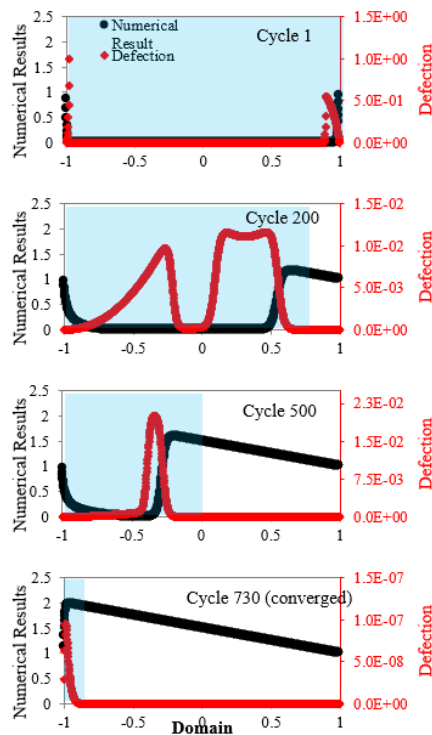


Figure 10. Convergence behaviors of Case D

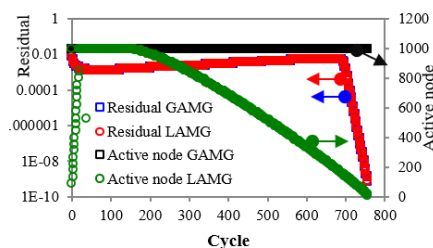


Figure 11. Active nodes relative to residual declination on Case D

As result in Table 1, it was notice that (a) both methods, LAMG and GAMG, converged by the same cycles. It was possible to conclude that both methods were state on same accuracy. (b) Because of the LAMG method was the localized technique, it was going to be that the LAMG method used fewer time compared to the GAMG method. For the test cases could save up to 32, 50, 22 and 23 percent for case A, B, C and D, respectively. Even non-linear problems such a case D can be solved by the LAMG method. This demonstrates computational stability over the varied and mathematically complex problems of the LAMG method. And (c) the saved time, on the same case, has been trained to be saver when computational nodes increased.

Table 1. Summary result of all 1D cases

Case	Node	GAMG		LAMG		Time Save (%)
		Cycle	Time (second)	Cycle	Time (second)	
A	1000	492	1.26	492	1.02	19.05
	2000	1499	6.71	1499	4.98	25.78
	4000	5368	44.29	5368	29.84	32.63
	8000	19875	308.14	19875	207.47	32.67
B	1000	86	0.3	86	0.22	26.67
	2000	228	1.13	228	0.68	39.82
	4000	651	5.62	651	2.9	48.40
	8000	1996	31.62	1996	15.69	50.38
C	1000	3647	9.77	3647	8.5	13.00
	2000	11992	54.14	11992	43.7	19.28
	4000	37934	318.85	37934	247.69	22.32
	8000	115041	1839.49	115041	1441.26	21.65
D	1000	730	2.27	730	1.96	13.66
	2000	2854	14.82	2854	11.93	19.50
	4000	11263	105.16	11263	82.56	21.49
	8000	44641	788.16	44641	601.91	23.63

The performing on the Poisson's equation and the boundary condition was evaluated. The results of the three distinct numerical techniques: single grid technique, multigrid technique, and the local multigrid technique, were taken to compare with the exact solution (11). The solutions on cell center relate to the x-y coordinate is shown in Figure 12. The figure shown that all the three numerical results are trend along to each other's and correlated with the exact solution. Moreover, the numerical errors were lower than 5% and corresponded with each other as well. It is obvious that the accuracy of the numerical results is reliable.

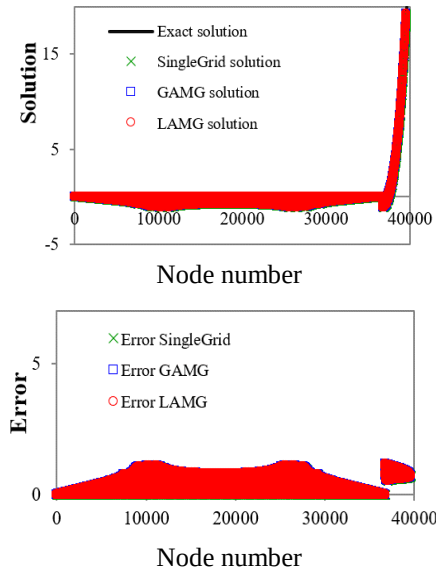


Figure 12. The evaluation of the accuracy on case 2D

The declining of the error and active nodes which compared between GAMG and LAMG technique, on the same iteration, are shown in Figure 13. The developing of the errors was represented with the green shades point out to green y-axis (left), meanwhile the blue y-axis (right) referred to active nodes was identifying by blue shades at the same iteration (x-axis). Zero was used for initialization. At first, the LAMG method employed the same amount of grid as the GAMG method. Until the 100th iteration, the LAMG method used decreased grids. Shows that the active zone has been scaled down, indicating that the computations have begun to partially converge. However, the reduction in the error (green) compared to both GAMG and LAMG method, there was a corresponding decrease sequentially until convergence.

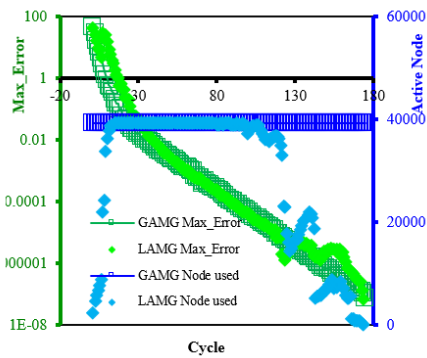


Figure 13. Active nodes relative to residual declination on Case 2D

Consider the adaptation of the active zone while the LAMG was computing, rendered in cycles 100, 120, 140, and 160, as shown in Figure 14 respectively. The active zone was a single continuous area that was not separated from each other. When the computing began, the active zone (grey) was the entire area and the area was scaled down as the computation started to converge. While the mature zones, which had an under-threshold tolerance, were the white regions. The adaptation in the active zone was unpredictable. They can also be expanded in size. It has the advantage of illustrating the program's calculating flexibility while also reflecting the lack of computational stability.

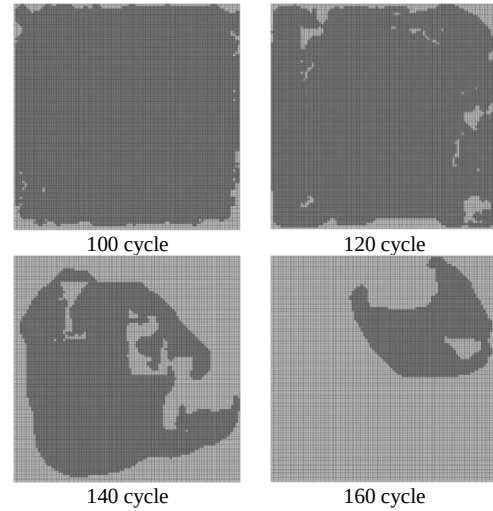


Figure 14. The adaptations of the active zone

However, as demonstrated in table 2, the technique of improving numerical area can decrease computational time. Examples of calculation results are displayed on three grid sizes: 200×200 , 400×400 , and 600×600 elements. The results demonstrated that the LAMG technique can save 6 to 14% less computing time than the GAMG technique for the test case. Furthermore, it can be seen that, whereas the LAMG technique used a greater number of iterations in some cases, but with the advantage of the localized technique was that it can converged faster than the global technique.

Table 2. Summary result of Case 2D

Node	GAMG		LAMG		Time Save (%)
	Cycle	Time (second)	Cycle	Time (second)	
200×200	175	15.85	174	13.53	14.64
400×400	264	120.15	271	103.19	14.12
600×600	187	203.3	223	190.73	6.18

Conclusions

The approach combines the advantages of the AMG method, which is highly flexible, together with the adaptive region technique called the LAMG method. This is a truly localized technique. The multigrid technique effectively eliminates low-frequency mode error, which is the primary cause of convergence delays. Adaptive region technique creates a computational environment that occurs only in the needed areas in order to reduce the computational work, called active zone. Defection is used to define the active zone, which is the only region where the multigrid is being used. Preliminary, the efficiency of the LAMG method was evaluated against 1-dimensional boundary layer problems, both linear and non-linear. It was found that the LAMG method was calculated as accurate as the GAMG method, but at significantly faster convergence rates. This supports the assumption that convergence is a regional property and so it is possible to calculate only certain areas that are significant without affecting the accuracy of the results. The computed results in 2D problems, the LAMG method can converge faster than the GAMG method, although in some cases it takes more computational cycles, showing the advantage of the method. The use of higher computational cycles of the LAMG method reflects that the lack of computational stability. This is necessary for further improvements in computational stability. This will result in an increase in computational efficiency to reduce the computation time further.

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