

FINITE BEAM ELEMENT BASED ON THE STRAIN-DRIVEN MODEL FOR ANALYSES OF NANOBAMS ON SUBSTRATE MEDIA

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Abstract

The nonlocal elasticity theory has been widely employed to represent the phenomena in small-scale effects, in micro/nano-structures. For the planar Euler-Bernoulli beam problem, the differential form of this nonlocal theory is preferable for use in conjunction with the classical beam model, due to its simple features. However, this form leads to paradoxical results in some cases. Therefore, this paper aimed to develop and propose a beam model to resolve the inconsistent responses via the integral form of nonlocal elasticity theory, for the analysis of nanobeam on substrate medium. The Gurtin-Murdoch surface model and the Winkler-foundation model are used to incorporate size-dependent and substrate-structure interaction effects into the nonlocal beam model. The proposed model is constructed based on the displacement-based finite element formulation. To demonstrate the capability of the novel proposed model and to characterize bending responses, three numerical simulations are used. The first simulation proves the model's competence to overcome the inconsistent results. The second simulation demonstrates the impact of the nonlocal and surface-energy effects in the bending responses of nanobeam on the elastic substrate medium. Finally, the third simulation examines the influences of several model parameters on contact stiffness.

Keywords: Nanobeam, Eringen nonlocal elasticity theory, Strain-driven model, Integral constitutive equation, Surface-energy effect, Elastic substrate media

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Introduction

With the current developments in modern technology, nanoscale materials and structures are of interest to pioneering engineers dealing with the design requirements of several applications (Iijima and Ichihashi, 1993; Calvert *et al.*, 2015; Jamali *et al.*, 2016; Salamat and Sedighi, 2017; Malikan and Eremeyev, 2021; Zhang *et al.*, 2021), involving carbon nanotubes (CNTs), nanoelectromechanical systems (NEMS), and nanocomposite materials. In depth understanding of the mechanical behaviors and the miniaturized structure models (e.g. bar, beam, shell, and plate) is required. This motivates the development of well-posed models for the study of these behaviors in several research works, including this current one. It is well-known that the classical models based on elasticity theory are insufficient to characterize the sized-scale phenomena due to the lack of an internal material length parameter, as recognized in the literature (Papargyri-Beskou *et al.*, 2003; Gonella *et al.*, 2011; Carcaterra, 2015; Civalek and Numanoglu, 2020; Malikan and Eremeyev, 2020; Sae-Long *et al.*, 2020). In general, there are two alternative models in the engineering fields to represent the size effects within the micro/nanoscale structures, namely atomistic models (Coluci *et al.*, 2007; Puligheddu and Galli, 2020; Kateb, 2020; Wang *et al.*, 2021) and continuum models (Limkatanyu *et al.*, 2014; Barretta and Marotti de Sciarra, 2019; Barretta *et al.*, 2019; Maneshi *et al.*, 2020; Pei and Li, 2021). The atomistic models can represent the mechanical responses of the nanostructures well. However, they suffer from high computational cost and complexity in the formulation (Coluci *et al.*, 2007; Kateb, 2020; Puligheddu and Galli, 2020; Wang *et al.*, 2021). Therefore, several studies have focused on the continuum models that compromise between accuracy and simplicity. Anyway, the conventional continuum model is not appropriate to characterize the nanostructures as mentioned above. Thus, higher-order non-classical theories are employed to overcome the limitations of classical models. These include, for example, the nonlocal elasticity theory (Eringen, 1972, 1983), strain gradient theory (Mindlin, 1965), couple stress theory (Mindlin, 1963; Toupin, 1964), modified strain gradient theory (Lam *et al.*, 2003), and modified couple stress theory (Yang *et al.*, 2002).

Among the higher-order non-classical theories listed above, the nonlocal elasticity theory of Eringen (Eringen, 1972, 1983) has attracted significant interest in the last two decades, and is widely employed to study both static and dynamic

analyses of nanobeams (Juntarasaid *et al.*, 2012; Eptaimeros *et al.*, 2016; Limkatanyu *et al.*, 2018; Ebrahimi *et al.*, 2020; Malikan and Eremeyev, 2020; Arif and Lellep, 2022). This theory was first introduced by Kröner (Kröner, 1967) and was later formally compiled by Eringen (Eringen, 1972). The original form of this theory was expressed in integral form, in which a kernel function was employed to represent the small-scale effects. Later, Eringen (Eringen, 1983) converted the integral form into a differential form based on a specific kernel function. Due to its simplicity, the differential form of Eringen nonlocal elasticity theory (Eringen, 1983) has been extensively employed to study the size-scale effects of nanostructures (Juntarasaid *et al.*, 2012; Eptaimeros *et al.*, 2016; Limkatanyu *et al.*, 2018; Ebrahimi *et al.*, 2020; Malikan and Eremeyev, 2020; Arif and Lellep, 2022). However, there are several questionable and paradoxical results in some cases of bending analysis of the cantilever nanobeam subjected to free-end concentrated load (Peddieson *et al.*, 2003; Challamel *et al.*, 2014; Khodabakhshi and Reddy, 2015; Ceballes *et al.*, 2021; Sae-Long *et al.*, 2021). For instance, the influence of small-scale effects vanished when the nonlocal parameter associated with the long-range inter-atomic interaction increased. This issue was later explained by Challamel *et al.* (Challamel *et al.*, 2014) as follows: the transformation from integral form into the differential form as proposed by Eringen (Eringen, 1983) was a non-injective process in the finite domain. Furthermore, Romano *et al.* (Romano *et al.*, 2017) noted that this inconsistent result was caused by the incompatibility between the equilibrium conditions and constitutive boundary conditions. To remedy these limitations, Challamel *et al.* (Challamel *et al.*, 2014) and Zhang (Zhang, 2017) suggested two alternative approaches. The first approach aims to establish a functional through an inverse procedure, while the second way is by using the integral form of the Eringen nonlocal elasticity theory.

In the second approach based on using the Eringen nonlocal integral model, the “strain-driven” nonlocal model has been extensively used to study the bending behavior of nanobeam (Fernández-Sáez *et al.*, 2016; Koutsoumaris *et al.*, 2017; Norouzzadeh and Ansari, 2017; Oskouie *et al.*, 2018). For example, Fernández-Sáez *et al.* (Fernández-Sáez *et al.*, 2016) presented the differences between the Eringen nonlocal integral model and its differential counterpart of the static bending problem. Norouzzadeh and Ansari (Norouzzadeh and Ansari, 2017) employed the finite element method to overcome the difficulty

of using the Eringen nonlocal integral model for the Timoshenko nanobeam problem. Koutsoumaris *et al.* (Koutsoumaris *et al.*, 2017) developed the two-phase nonlocal integral (TPNI) model to investigate the static responses of Euler-Bernoulli nanobeam with various loading and boundary conditions. Oskouie *et al.* (Oskouie *et al.*, 2018) compared the difference between strain-and stress-driven integral models for bending analysis of Euler-Bernoulli nanobeams. All the literature cited above (Fernández-Sáez *et al.*, 2016; Koutsoumaris *et al.*, 2017; Norouzzadeh and Ansari, 2017; Oskouie *et al.*, 2018) confirms the capability of the Eringen nonlocal integral model to overcome the paradoxical responses of the bending analysis for nanobeam problems.

In addition to the small-scale effects, the size-dependent behavior plays a crucial role in the mechanical responses of nanostructures when the surface area to volume ratio becomes large (Intarit *et al.*, 2011, 2018). In order to consider the size-dependent effects within the classical continuum models, Gurtin and Murdoch (Gurtin and Murdoch, 1975, 1978) presented their surface elasticity theory for isotropic materials. Their theory has been extensively employed to study the influences of surface energy and its residual stresses in the bending analysis of nanostructure problems (Liu *et al.*, 2011; Juntarasaid *et al.*, 2012; Limkatanyu *et al.*, 2018; Lu *et al.*, 2021; Sae-Long *et al.*, 2021). For instance, Juntarasaid *et al.* (Juntarasaid *et al.*, 2012) and Limkatanyu *et al.* (Limkatanyu *et al.*, 2018) proposed the Eringen nonlocal differential beam model with the inclusion of surface effects based on the Gurtin and Murdoch continuum model (Gurtin and Murdoch, 1975, 1978) to study the bending responses of nanobeam, while Sae-Long *et al.* (Sae-Long *et al.*, 2021) improved their models (Juntarasaid *et al.*, 2012; Limkatanyu *et al.*, 2018) by replacing the Eringen nonlocal model with the thermodynamics-based nonlocal Barretta and Marotti de Sciarra (Barretta and Marotti de Sciarra, 2013) model to account for the small-scale effects within nanobeam model. Liu *et al.* (Liu *et al.*, 2011) incorporated the surface-energy effects within the finite non-classical beam model to investigate the size dependence of static, dynamic, and buckling analysis of the nanobeam problem. Lu *et al.* (Lu *et al.*, 2021) modified the surface elasticity theory of Gurtin and Murdoch (Gurtin and Murdoch, 1975, 1978) based on the equilibrium equation on the surface for the study of the influence of the surface effect on the static bending problem for the simply supported nanobeam. All these studies have demonstrated the significance of the surface phenomena in the bending analysis of nanobeams. Furthermore, they have affirmed the capability of

the Gurtin and Murdoch surface model (Gurtin and Murdoch, 1975, 1978) to represent the surface-energy effects within the nano-scale continuum beam model.

Nanostructure on the substrate medium is a fundamental problem, which has been found in nanoengineering and nanoscience applications, such as electronic logic gates, flexible plastic electronics (Hsieh *et al.*, 2008), biological sensing (Wang and Gates, 2009), electronic devices reinforced in polymer nanocomposites (Park *et al.*, 2008), and stretchable electronics (Qi *et al.*, 2017). To provide in-depth understanding of the mechanical characteristics for design of such applications, several researchers (Shen and Zhang, 2011; Khajeansari *et al.*, 2012; Ponbunyanon *et al.*, 2016; Malikan *et al.*, 2017; Demir *et al.*, 2018; Pour *et al.*, 2021) have attempted to develop the mathematical models under the concept of the beam on foundation system for the study of static responses in the nanostucture problem on the substrate medium. For example, Shen and Zhang (Shen and Zhang, 2011) formulated the single-wall carbon nanotubes (SWCNTs) model on a two-parameter foundation in thermal environments for the nonlinear bending, buckling, and vibration analysis of nanoscale structures. Khajeansari *et al.* (Khajeansari *et al.*, 2012) developed and proposed explicit solutions for the static bending responses of the nanowire problems resting on the Winkler-Pasternak foundation with inclusion of the surface effects. Ponbunyanon *et al.* (Ponbunyanon *et al.*, 2016) extended the work of Limkatanyu *et al.* (Limkatanyu *et al.*, 2015a) to propose the force-based beam-elastic substrate model to investigate the influence of small-scale and size-dependent effects on the bending responses. The interaction between nanobeam and substrate medium of their model (Ponbunyanon *et al.*, 2016) was represented through assumption of the Winkler foundation model. Malikan *et al.* (Malikan *et al.*, 2017) investigated the shear and thermal buckling for the bi-layer orthotropic nanoplate resting on the Winkler-Pasternak foundation. Demir *et al.* (Demir *et al.*, 2018) studied the static bending responses in parametric studies of their proposed nanobeam model on the Winkler foundation (Winkler, 1867). Pour *et al.* (Pour *et al.*, 2021) studied the thermal buckling characteristic of the circular bilayer orthotropic nanoplate embedded in the Winkler-Pasternak elastic medium. All the literature cited above applied the differential model of Eringen nonlocal elasticity theory to account for the nonlocality within their models (Shen and Zhang, 2011; Khajeansari *et al.*, 2012; Ponbunyanon *et al.*, 2016; Malikan *et al.*, 2017; Demir *et al.*, 2018; Pour *et al.*, 2021). However, when the interaction

between the substrate medium and nanostructures disappears, inconsistencies and paradoxes may arise in these models for cantilever-supported structure problems subjected to free-end point load (Sae-Long *et al.*, 2021).

The motivation of the current paper was to develop a “*paradox-free*” nanobeam-elastic substrate medium model for bending analysis of the Euler-Bernoulli nanobeam problems resting on the substrate medium. The Eringen nonlocal integral model is used to incorporate the small-scale effects within the nanobeam model, while the surface continuum model of Gurtin and Murdoch (Gurtin and Murdoch, 1975, 1978) and the Winkler-foundation model are employed to represent the size-dependent and substrate-structure interaction effects, respectively. The proposed model is formulated in the displacement-based finite element formulation. All procedures in this paper are implemented in the finite element platform FEAP (Taylor, 2000) together with the Mathematica computer software (Wolfram, 1992).

This paper is structured as follows: The fundament model and its assumptions employed to characterize the behaviors of the nanobeam model on the substrate medium are first presented. It includes the strain-driven integral elasticity model (Oskouie *et al.*, 2018), surface-energy continuum model (Gurtin and Murdoch, 1975, 1978), and substrate medium model (Limkatanyu *et al.*, 2013). Next, this section introduces the displacement-based formulation of the proposed model. In this section, the governing differential equation and its associated natural boundary conditions are derived based on the principle of virtual displacement. From the weak form of the system equilibrium, a variational method is employed to establish the element stiffness of the proposed model. The integral term of the sectional nonlocal bending moment corresponding to the Eringen nonlocal integral model (Oskouie *et al.*, 2018) can be determined from the iteration procedure elaborated in this section. Subsequently, this section demonstrates the numerical simulations to characterize the nanobeam systems. Finally, the last section presents the conclusion of the overall content and observations in this paper.

Fundamental Model and Its Assumptions for Nanobeam on the Elastic Substrate Medium

To establish the non-classical beam-substrate model for analysis of the nanobeam problem resting on the elastic substrate medium requires that the theories, assumptions, and mathematical models incorporate the nonlocality, the surface-energy, and the substrate-structure interaction effects within the classical beam model. In this section, the

fundamental models, which are employed to represent these effects, are briefly discussed, namely the strain-driven integral elasticity model, the surface-energy continuum model, and the substrate medium model.

Strain-driven integral elasticity model

For the conventional structural models, the stress-strain relation for the linear elastic and homogeneous isotropic material based on the elasticity theory can be expressed as (Oskouie *et al.*, 2018):

$$\sigma_{ij}^L(x) = \lambda \delta_{ij} \varepsilon_{kk}(x) + 2\mu \varepsilon_{ij}(x) \quad (1)$$

where $\sigma_{ij}^L(x)$ is the conventional (local) stress tensor together with its conjugate work pair $\varepsilon_{ij}(x)$, and λ and μ are the Lamé constants; and the subscript indexes i , j , and k each indicate the direction of any coordinate axis.

To define the nonlocal effects within the conventional elasticity theory, the nonlocal elasticity theory of Eringen (Eringen, 1972) is employed here. This theory implies that the stress at the particular point x of the domain is not only a function of the strain at that point but also a function of the strains at other points. In their assumption, the non-classical (nonlocal) stress tensor $\sigma_{ij}^N(x)$ at point x of the domain can be expressed in an integral form as follows:

$$\sigma_{ij}^N(x) = \int_{\bar{x}} k(|x - \bar{x}|, e_0 a) \sigma_{ij}^L(\bar{x}) d\bar{x} \quad (2)$$

where $\sigma_{ij}^N(x)$ is the nonlocal stress tensor; k is the kernel function; $|x - \bar{x}|$ is the neighborhood distance; $e_0 a$ is the material length-scale (nonlocality) parameter; and $\bar{x}$ is the reference point. To conveniently define the reference point $\bar{x}$, this work assigns it at the classical Gauss point (Chapra and Canale, 2015) in each element.

Based on the nonlocal constitutive Equation (2), this relation can be rewritten into Equation (3) for the planar Euler-Bernoulli beam problem (Oskouie *et al.*, 2018):

$$\sigma_{xx}^N(x) = \int_{\bar{x}} k(|x - \bar{x}|, e_0 a) E_{xx} \varepsilon_{xx}(\bar{x}) d\bar{x} \quad (3)$$

where E_{xx} is the beam bulk modulus.

According to the kinematic assumption of the Euler-Bernoulli beam theory, the plane section is

assumed to remain planar, and normal to the beam axis after deformation. Thus, the displacement fields in x -, y -, and z - components of this theory can be introduced as:

$$U_x(x) = -y \frac{dv(x)}{dx}; U_y(x) = v(x); \text{ and } U_z(x) = 0 \quad (4)$$

where $U_x(x)$, $U_y(x)$, and $U_z(x)$ are the displacement fields in x -, y -, and z -component, respectively; y is the generic distance in y -component; and $v(x)$ is the transverse displacement of nanobeam.

The axial strain $\varepsilon_{xx}(x)$ relates to the axial displacement $U_x(x)$ through the compatibility relation as follows:

$$\varepsilon_{xx}(x) = \frac{dU_x(x)}{dx} = -y\kappa(x) = -y \frac{d^2v(x)}{dx^2} \quad (5)$$

where $\kappa(x) = d^2v(x)/dx^2$ is the nanobeam curvature.

Surface-energy continuum model

In a case with a high surface-to-bulk ratio, the influence of the different energies of atoms on the free surface and in the material bulk plays an essential role in the micro/nanostructure responses (Intarit *et al.*, 2011, 2018). This phenomenon is named the “surface-energy” effect. To incorporate this effect within the classical continuum models, several studies (Gurtin and Murdoch, 1975, 1978; Intarit *et al.*, 2011, 2018; Liu *et al.*, 2011; Lu *et al.*, 2021; Wongviboonsin *et al.*, 2021) have attempted to develop and propose mathematical models. Among these models, the Gurtin and Murdoch continuum surface model (Gurtin and Murdoch, 1975, 1978) is among the most widely used ones in the engineering community, due to its simplicity. Under their assumptions the outer surface was defined as a mathematically-zero-thickness layer, while the bond between beam bulk and the outer surface layer was perfect based on the surface theory of Cammarata (Cammarata, 1994), as shown in Figure 1. Therefore, the surface constitutive relations are

$$\tau_{\alpha\beta}^{sur} = \left[\tau_0^{sur} + \left(\lambda_0^{sur} + \mu_0^{sur} \right) u_{\gamma,\gamma}^{sur} \right] \delta_{\alpha\beta} + \mu_0^{sur} \left(u_{\alpha,\beta}^{sur} + u_{\beta,\alpha}^{sur} \right) - \tau_0^{sur} u_{\beta,\alpha}^{sur} \quad (6)$$

$$\tau_{n\alpha}^{sur} = \tau_0^{sur} u_{n,\alpha}^{sur} \quad (7)$$

where $\tau_{\alpha\beta}^{sur}$ and $\tau_{n\alpha}^{sur}$ are the in-plane and out-of-plane components of the surface stress tensor, respectively; α and β are the in-plane Cartesian coordinates of beam lateral surface ($\alpha, \beta = 1, 2$); τ_0^{sur} is the residual surface stress under unconstrained conditions obtained from an atomistic simulation (Miller and Shenoy, 2000); u^{sur} is the surface deformation; and $\delta_{\alpha\beta}$ is the Kronecker delta.

The non-zero surface components in Equation (6) and Equation (7) for the planar Euler-Bernoulli nanobeams (Gao and Mahmoud, 2014) are

$$\tau_{xx}^{sur}(x) - \tau_0^{sur} = E_{xx}^{sur} \varepsilon_{xx}^{sur}(x) \quad (8)$$

$$\tau_{nx}^{sur}(x) = \tau_0^{sur} \varepsilon_{nx}^{sur}(x) \quad (9)$$

where τ_{xx}^{sur} and τ_{nx}^{sur} are the surface stress in-plane and out-of-plane components, respectively; ε_{xx}^{sur} and ε_{nx}^{sur} are the surface strain in-plane and out-of-plane components, respectively; and E_{xx}^{sur} is the elastic modulus of surface layer.

Based on the surface elasticity theory (Gao and Mahmoud, 2014), the compatibility relations of the surface-layer can be expressed as:

$$\varepsilon_{xx}^{sur}(x) = \varepsilon_{xx}(x) = \frac{dU_x(x, y)}{dx} \quad (10)$$

and $\varepsilon_{nx}^{sur}(x) = n_y \frac{dU_y(x)}{dx}$

where n_y is y -component of the unit vector n that is normal to the lateral surface of beam section, as shown in Figure 1.

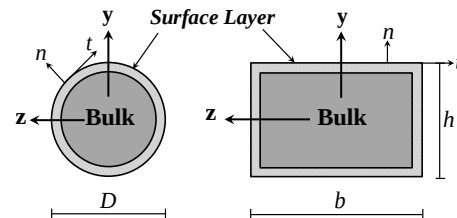


Figure 1. Beam cross section with the surface layer (Sae-Long *et al.*, 2021)

Substituting the beam kinematic assumption of Equation (4) into Equation (10) and subsequently placing these relations into Equation (8) and Equation (9), the non-zero surface stresses (τ_{xx}^{sur} and

τ_{xx}^{sur}) in terms of the transverse displacement $v(x)$ can be written as:

$$\tau_{xx}^{sur}(x) - \tau_0^{sur} = -y E_{xx}^{sur} \kappa^{sur}(x) \quad (11)$$

$$\tau_{xx}^{sur}(x) = \tau_0^{sur} n_y \frac{dv(x)}{dx} \quad (12)$$

It is worth noting from Equation (11) that the surface bending curvature $\kappa^{sur}(x) = d^2v(x)/dx^2$ is compatible with the beam curvature $\kappa(x) = d^2v(x)/dx^2$ following the perfect bond assumption of surface elasticity theory (Cammarata, 1994).

Substrate medium model

This study employs the concept of a beam on an elastic foundation (Limkatanyu *et al.*, 2013) to represent the interaction effects between nanobeam and substrate medium. Based on the fundamental hypothesis of the Winkler foundation model (Winkler, 1867), the underlying substrate medium is replaced with a set of continuously distributed non-interconnected springs attached along the beam length. Therefore, the constitutive relation of the substrate medium can be written in the following form:

$$D_s(x) = k_s \Delta_s(x) \quad (13)$$

where $D_s(x)$, $\Delta_s(x)$, and k_s are the substrate interaction force, substrate deformation, and elastic substrate stiffness, respectively. Following the concept of the Winkler foundation model (Limkatanyu *et al.*, 2013), the bond between the nanobeam and the substrate media is assumed to be perfect. Thus, their compatibility relation can be written as:

$$\Delta_s(x) = v(x) \quad (14)$$

Formulation

Governing differential equilibrium equation and its boundary conditions

In the present work, the governing differential equilibrium equation and its boundary condition for the nanobeam on the elastic substrate medium, as illustrated in Figure 2, are derived based on the principle of virtual displacement. The general form of virtual work is

$$\delta W = \delta W_{int} + \delta W_{ext} = 0 \quad (15)$$

where δW , δW_{int} , and δW_{ext} are the system total, internal, and external virtual works, respectively.

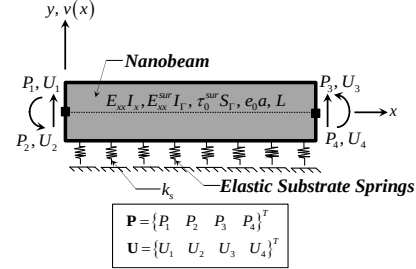


Figure 2. Proposed nanobeam-elastic substrate element

The system internal and external virtual works for the proposed model can be expressed as:

$$\delta W_{int} = \int \left(\int \sigma_{xx}^N(x) dA \right) \delta \varepsilon_{xx}(x) dx + \int D_s(x) \delta \Delta_s(x) dx \quad (16)$$

$$+ \int \left(\left(\tau_{xx}^{sur}(x) - \tau_0^{sur} \right) \delta \varepsilon_{xx}^{sur}(x) \right) d\Gamma dx + \int \left(\tau_{xx}^{sur}(x) \delta \varepsilon_{xx}^{sur}(x) \right) d\Gamma dx$$

$$\delta W_{ext} = - \int p_y(x) \delta v(x) dx - \mathbf{P}^T \mathbf{U} \quad (17)$$

where L is the nanobeam length; A is nanobeam section area; Γ is the nanobeam section perimeter; $p_y(x)$ is the applied transverse load; and $\mathbf{U} = \{U_1, U_2, U_3, U_4\}^T$ is the displacement vector containing the nanobeam end displacements, which correspond to the nanobeam end forces collected in the force vector $\mathbf{P} = \{P_1, P_2, P_3, P_4\}^T$ as illustrated in Figure 2.

Substituting the compatibility relations of Equations (5), (10), and (14) into Equation (16) results in

$$\delta W_{int} = \int_L M^N(x) \frac{d^2 \delta v(x)}{dx^2} dx + \int_L D_s(x) \delta v(x) dx + \int_L M_{\tau_{xx}^{sur}-\tau_0^{sur}}^{sur}(x) \frac{d^2 \delta v(x)}{dx^2} dx + \int_L V_{\tau_{xx}^{sur}}^{sur}(x) \frac{d \delta v(x)}{dx} dx \quad (18)$$

where $M^N(x)$ is the nonlocal bending moment; $M_{\tau_{xx}-\tau_0}^{sur}(x)$ is the surface sectional moment; and $V_{\tau_{xx}}^{sur}(x)$ is the surface sectional shear force. These sectional resultant forces can be defined as follows:

$$M^N(x) = -\int_A y \sigma_{xx}(x) dA \quad (19)$$

$$M_{\tau_{xx}-\tau_0}^{sur}(x) = -\int_{\Gamma} \tilde{h}_y (\tau_{xx}^{sur}(x, y) - \tau_0^{sur}) d\Gamma \quad (20)$$

$$V_{\tau_{xx}}^{sur}(x) = \int_{\Gamma} \tilde{h}_y \tau_{xx}^{sur}(x) d\Gamma \quad (21)$$

With the relations in Equation (17) and Equation (18), the total virtual work in Equation (15) can be expressed as follows:

$$\begin{aligned} \delta W = & \int_L M^N(x) \frac{d^2 \delta v(x)}{dx^2} dx \\ & + \int_L M_{\tau_{xx}-\tau_0}^{sur}(x) \frac{d^2 \delta v(x)}{dx^2} dx \\ & + \int_L V_{\tau_{xx}}^{sur}(x) \frac{d \delta v(x)}{dx} dx \\ & + \int_L D_s(x) \delta v(x) dx \\ & - \int_L p_y(x) \delta v(x) dx - \delta \mathbf{U}^T \mathbf{P} = 0 \end{aligned} \quad (22)$$

Applying integration by parts to Equation (22), to move the differential operators from the nanobeam deflection to the sectional resultant forces, results in:

$$\begin{aligned} \delta W = & \int_L \delta v(x) \left[\frac{d^2 M^N(x)}{dx^2} + \frac{d^2 M_{\tau_{xx}-\tau_0}^{sur}(x)}{dx^2} - \frac{dV_{\tau_{xx}}^{sur}(x)}{dx} + D_s(x) - p_y(x) \right] dx \\ & + \delta v(x) \left[\frac{dM^N(x)}{dx} - \frac{dM_{\tau_{xx}-\tau_0}^{sur}(x)}{dx} + V_{\tau_{xx}}^{sur}(x) \right] \Big|_{x=0}^{x=L} \\ & + \frac{d \delta v(x)}{dx} \left[\frac{M^N(x) + M_{\tau_{xx}-\tau_0}^{sur}(x)}{M(x)} \right] \Big|_{x=0}^{x=L} - \delta \mathbf{U}^T \mathbf{P} = 0 \end{aligned} \quad (23)$$

It needs to be emphasized regarding Equation (23) that the sectional bending moment $M(x)$ is contributed not only by the nanobeam bending moment $M^N(x)$ but also by the surface-induced distributed moment $M_{\tau_{xx}-\tau_0}^{sur}(x)$. Like the sectional bending moment, the total sectional shear force

$V(x)$ is contributed not only by the derivative of nanobeam bending moment $dM^N(x)/dx$ but also by the derivative of the surface-induced distributed moment $dM_{\tau_{xx}-\tau_0}^{sur}(x)/dx$ and by the surface-induced shear force $V_{\tau_{xx}}^{sur}(x)$. Therefore, both the sectional bending moment $M(x)$ and the total sectional shear force $V(x)$ can be defined as:

$$M(x) = M^N(x) + M_{\tau_{xx}-\tau_0}^{sur}(x) \quad (24)$$

$$V(x) = -\frac{dM^N(x)}{dx} - \frac{dM_{\tau_{xx}-\tau_0}^{sur}(x)}{dx} + V_{\tau_{xx}}^{sur}(x) \quad (25)$$

Following the Cartesian sign convention, the relation in Equation (23) can be expressed as follows:

$$\begin{aligned} \delta W = & \int_L \delta v(x) \left[\frac{d^2 M^N(x)}{dx^2} + \frac{d^2 M_{\tau_{xx}-\tau_0}^{sur}(x)}{dx^2} - \frac{dV_{\tau_{xx}}^{sur}(x)}{dx} + D_s(x) - p_y(x) \right] dx \\ & - \delta U_1 \left[P_1 + \left(-\frac{dM^N(x)}{dx} - \frac{dM_{\tau_{xx}-\tau_0}^{sur}(x)}{dx} + V_{\tau_{xx}}^{sur}(x) \right) \right] \Big|_{x=0} \\ & - \delta U_2 \left[P_2 + (M^N(x) + M_{\tau_{xx}-\tau_0}^{sur}(x)) \right] \Big|_{x=0} \\ & - \delta U_3 \left[P_3 - \left(-\frac{dM^N(x)}{dx} - \frac{dM_{\tau_{xx}-\tau_0}^{sur}(x)}{dx} + V_{\tau_{xx}}^{sur}(x) \right) \right] \Big|_{x=L} \\ & - \delta U_4 \left[P_4 - (M^N(x) + M_{\tau_{xx}-\tau_0}^{sur}(x)) \right] \Big|_{x=L} = 0 \end{aligned} \quad (26)$$

Based on the relation in Equation (26), the governing differential equilibrium equation of the proposed model can be presented, by considering that $\delta v(x)$ was arbitrary, as:

$$\frac{d^2 M^N(x)}{dx^2} + \frac{d^2 M_{\tau_{xx}-\tau_0}^{sur}(x)}{dx^2} - \frac{dV_{\tau_{xx}}^{sur}(x)}{dx} + D_s(x) - p_y(x) = 0 \quad \text{for } x \in (0, L) \quad (27)$$

Since $\delta \mathbf{U}$ is also arbitrary in Equation (26), the natural boundary conditions are:

$$\begin{aligned} P_1 = & - \left[-\frac{dM^N(x)}{dx} - \frac{dM_{\tau_{xx}-\tau_0}^{sur}(x)}{dx} + V_{\tau_{xx}}^{sur}(x) \right] \Big|_{x=0} ; \\ P_2 = & - \left[M^N(x) + M_{\tau_{xx}-\tau_0}^{sur}(x) \right] \Big|_{x=0} ; \\ P_3 = & \left[-\frac{dM^N(x)}{dx} - \frac{dM_{\tau_{xx}-\tau_0}^{sur}(x)}{dx} + V_{\tau_{xx}}^{sur}(x) \right] \Big|_{x=L} ; \\ P_4 = & \left[M^N(x) + M_{\tau_{xx}-\tau_0}^{sur}(x) \right] \Big|_{x=L} \end{aligned} \quad (28)$$

In this section, the sectional resultant forces in Equation (27) and Equation (28) can be written in

terms of the nanobeam deflection $v(x)$ through the constitutive relation by enforcing the compatibility equations.

For the nanobeam and surface layer, the force-displacement relations are obtained by substituting the constitutive relations in Equations (3), (8) and (9) and the compatibility Equation (5) and Equation (10) into the sectional resultant forces in Equations (19), (20) and (21). We then have

$$M^N(x) = \int_L k(|x-\bar{x}|, e_0 a) E_{xx} I_x \frac{d^2 v(x)}{dx^2} d\bar{x} \quad (29)$$

$$M_{\tau_{xx}-\tau_0}^{sur}(x) = E_{xx}^{sur} I_\Gamma \frac{d^2 v(x)}{dx^2} \quad (30)$$

$$V_{\tau_{xx}}^{sur}(x) = \tau_0^{sur} S_\Gamma \frac{dv(x)}{dx} \quad (31)$$

where $I_x = \int_A y^2 dA$ is the second moment over area; $I_\Gamma = \int_\Gamma \mathbf{N}^2 d\Gamma$ is the second moment perimeter; and $S_\Gamma = \int_\Gamma n_y^2 d\Gamma$.

For the underlying substrate medium, the force-displacement relation is obtained by substituting the compatibility equation of Equation (14) into the constitutive relation of Equation (13). We have

$$D_s(x) = k_s v(x) \quad (32)$$

Imposing the force-displacement relations of Equations (29), (30), (31) and (32) the governing differential Equation (27) can be rewritten in terms of the nanobeam deflection $v(x)$ on the domain $x \in (0, L)$ as:

$$\begin{aligned} & \frac{d^2}{dx^2} \left[\int_L k(|x-\bar{x}|, e_0 a) E_{xx} I_x \frac{d^2 v(x)}{dx^2} d\bar{x} \right] \\ & + E_{xx}^{sur} I_\Gamma \frac{d^4 v(x)}{dx^4} - \tau_0^{sur} S_\Gamma \frac{d^2 v(x)}{dx^2} \\ & + k_s v(x) - p_y(x) = 0 \end{aligned} \quad (33)$$

Similarly, the natural boundary conditions become:

$$\begin{aligned} P_1 &= - \left[-\frac{d}{dx} \left(\int_L k(|x-\bar{x}|, e_0 a) E_{xx} I_x \frac{d^2 v(x)}{dx^2} d\bar{x} \right) \right]_{x=0} \\ & \quad - \left[E_{xx}^{sur} I_\Gamma \frac{d^3 v(x)}{dx^3} + \tau_0^{sur} S_\Gamma \frac{dv(x)}{dx} \right]_{x=0} ; \\ P_2 &= - \left[\left(\int_L k(|x-\bar{x}|, e_0 a) E_{xx} I_x \frac{d^2 v(x)}{dx^2} d\bar{x} \right) \right]_{x=0} \\ & \quad + \left[E_{xx}^{sur} I_\Gamma \frac{d^2 v(x)}{dx^2} \right]_{x=0} ; \\ P_3 &= - \left[-\frac{d}{dx} \left(\int_L k(|x-\bar{x}|, e_0 a) E_{xx} I_x \frac{d^2 v(x)}{dx^2} d\bar{x} \right) \right]_{x=L} \\ & \quad - \left[E_{xx}^{sur} I_\Gamma \frac{d^3 v(x)}{dx^3} + \tau_0^{sur} S_\Gamma \frac{dv(x)}{dx} \right]_{x=L} ; \\ P_4 &= \left[\left(\int_L k(|x-\bar{x}|, e_0 a) E_{xx} I_x \frac{d^2 v(x)}{dx^2} d\bar{x} \right) \right]_{x=L} \\ & \quad + \left[E_{xx}^{sur} I_\Gamma \frac{d^2 v(x)}{dx^2} \right]_{x=L} \end{aligned} \quad (34)$$

It is important to note that the selection of the kernel function $k(|x-\bar{x}|, e_0 a)$ must be done with care due to potential paradoxical responses (Fernández-Sáez *et al.*, 2016; Koutsoumaris *et al.*, 2017; Norouzzadeh and Ansari, 2017; Oskouie *et al.*, 2018). In this work, the bi-exponential kernel function in Equation (35) was selected, as this function is the most widely employed and satisfies reasonable conditions, as suggested by Koutsoumaris *et al.* (Koutsoumaris *et al.*, 2017) and Oskouie *et al.* (Oskouie *et al.*, 2018). For instance, the kernel function converges to the Dirac delta function when the nonlocal parameter approaches zero, it is symmetric with respect to x , and the nonlocal model should approximate atomic lattice dynamics when the nonlocal parameter has a small value, etc.

$$k(|x-\bar{x}|, e_0 a) = \frac{1}{2e_0 a} e^{-\frac{|x-\bar{x}|}{e_0 a}} \quad (35)$$

According to the procedure to analyze the proposed nanobeam on the elastic substrate medium incorporating the surface-energy effect, it can fit the big picture based on Tonti's diagram (Tonti, 1976) shown in Figure 3(a). This diagram will be later modified to represent the displacement-based approach in the next topic.

Finite discretization

In the generalized displacement-based finite formulation, the nodal displacement $\mathbf{U}$ is treated as the primary element variable. Therefore, the transverse displacement $v(x)$ can be approximated

from the nodal displacement $\mathbf{U}$ through the displacement interpolation functions $\mathbf{N}(x)$ as follows:

$$v(x) = \mathbf{N}(x)\mathbf{U} \quad (36)$$

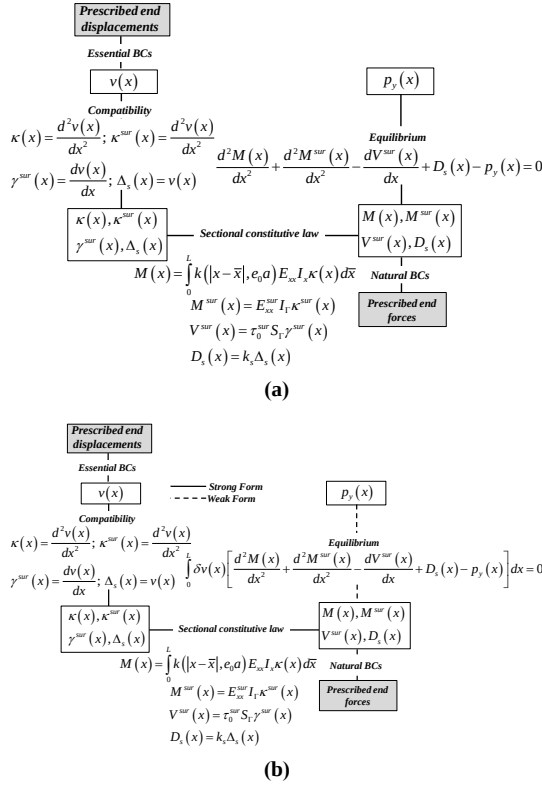


Figure 3. Tonti's diagrams for the displacement-based nanobeam model on an elastic substrate medium with inclusion of the surface-energy effect: (a) strong form; and (b) weak form

In this study, the displacement interpolation functions are approximated with cubic polynomials. With the differential operation, we can define:

$$\frac{dv(x)}{dx} = \frac{d\mathbf{N}(x)}{dx} \mathbf{U} = \mathbf{Y}(x)\mathbf{U}; \quad (37)$$

$$\text{and } \frac{d^2 v(x)}{dx^2} = \frac{d^2 \mathbf{N}(x)}{dx^2} \mathbf{U} = \mathbf{B}(x)\mathbf{U}$$

where $\mathbf{Y}(x)$ and $\mathbf{B}(x)$ form the deformation-displacement matrix.

The general framework of the displacement-based formulation for the proposed model is summarized in the modified Tonti's diagram of Figure 3(b). Based on the principle of virtual

displacement and kernel function of Equation (35), the weighted integral form of the system equilibrium equation is

$$\int_0^L \delta v(x) \left[\frac{d^2}{dx^2} \left[\int_0^L \frac{1}{2e_0 a} e^{-\frac{|x-\bar{x}|}{e_0 a}} E_{xx} I_x \frac{d^2 v(x)}{dx^2} d\bar{x} \right] + E_{xx}^{sur} I_\Gamma \frac{d^4 v(x)}{dx^4} - \tau_0^{sur} S_\Gamma \frac{d^2 v(x)}{dx^2} + k_s v(x) - p_y(x) \right] dx = 0 \quad (38)$$

The first three terms of Equation (38) are integrated by parts and we get

$$\begin{aligned} & \int_0^L \frac{d^2 \delta v(x)}{dx^2} E_{xx} I_x \left(\int_0^L \frac{1}{2e_0 a} e^{-\frac{|x-\bar{x}|}{e_0 a}} \frac{d^2 v(x)}{dx^2} d\bar{x} \right) dx \\ & + \int_0^L \frac{d^2 \delta v(x)}{dx^2} E_{xx}^{sur} I_\Gamma \frac{d^2 v(x)}{dx^2} dx \\ & + \int_0^L \frac{d \delta v(x)}{dx} \tau_0^{sur} S_\Gamma \frac{dv(x)}{dx} dx \\ & + \int_0^L \delta v(x) k_s v(x) dx \\ & = \delta \mathbf{U}^T \mathbf{P} + \int_0^L \delta v(x) p_y(x) dx \end{aligned} \quad (39)$$

The element stiffness equation is obtained by substituting Equation (36) into Equation (39) and due to the arbitrariness of $\delta \mathbf{U}$ this results in the following relation:

$$(\mathbf{K}_b + \mathbf{K}_s + \mathbf{K}_{sb} + \mathbf{K}_{ss}) \Delta \mathbf{U} = \mathbf{P} + \mathbf{P}_{py} \quad (40)$$

where $\mathbf{K}_b = E_{xx} I_x \int_0^L \mathbf{B}^T(x) \mathbf{B}^{eff}(x) dx$ is the nanobeam element stiffness; $\mathbf{K}_s = k_s \int_0^L \mathbf{N}^T(x) \mathbf{N}(x) dx$ is the substrate element stiffness; $\mathbf{K}_{sb} = E_{xx}^{sur} I_\Gamma \int_0^L \mathbf{B}^T(x) \mathbf{B}(x) dx$ and $\mathbf{K}_{ss} = \tau_0^{sur} S_\Gamma \int_0^L \mathbf{Y}^T(x) \mathbf{Y}(x) dx$ is the surface-layer element stiffness; $\mathbf{P}_{py} = \int_0^L \mathbf{N}^T(x) p_y(x) dx$ is the equivalent load vector; and $\mathbf{B}^{eff}(x) = \int_0^L \frac{1}{2e_0 a} e^{-\frac{|x-\bar{x}|}{e_0 a}} \mathbf{B}(\bar{x}) d\bar{x}$ is the effective deformation-displacement matrix.

The step-by-step algorithm to calculate the matrix $\mathbf{B}^{eff}(x_i^n)$ at each integration point i of the element n is shown in Figure 4, and can be expressed as follows:

1). Compute the deformation-displacement matrix $\mathbf{N}(x_i^n)$, $\mathbf{Y}(x_i^n)$, and $\mathbf{B}(x_i^n)$ at integration point i of the element n and get the sectional stiffness $E_{xx}I_x$, k_s , $E_{xx}^{sur}I_\Gamma$, and $\tau_0^{sur}S_\Gamma$.

2). Start the iterative procedure ($j \geq 1$) within the integration i of element n :

a. Compute the effective deformation-displacement matrix $\mathbf{B}^{eff}(x_i^n)$ as follows:

$$\mathbf{B}^{eff}(x_i^n) = \mathbf{B}^{eff}(x_i^n) + \sum_{j=1}^{NRP} w_j |J_j| \left(\frac{1}{2e_0 a} e^{\frac{|x_i^n - \bar{x}_j|}{e_0 a}} \mathbf{B}(\bar{x}_j) \right)$$

where w_j , J_j , and $\bar{x}_j$ are the weight function, the Jacobian, and the reference point j , respectively; and NRP is the number of the reference points.

3). Check the condition $j > NRP$:

a. If no, set $j = j + 1$ and return to step 2(a).

b. If yes, calculate the element stiffness and return its value.

$$(\mathbf{K})_i^n = (\mathbf{K})_i^n + \sum_{i=1}^{NIP} w_i |J_i| ((\mathbf{K}_b + \mathbf{K}_s + \mathbf{K}_{sb} + \mathbf{K}_{ss})_i^n)$$

With

$$(\mathbf{K}_b)_i^n = E_{xx} I_x \mathbf{B}^T(x_i^n) \mathbf{B}^{eff}(x_i^n)$$

$$(\mathbf{K}_s)_i^n = k_s \mathbf{N}^T(x_i^n) \mathbf{N}(x_i^n)$$

$$(\mathbf{K}_{sb})_i^n = E_{xx}^{sur} I_\Gamma \mathbf{B}^T(x_i^n) \mathbf{B}(x_i^n)$$

$$(\mathbf{K}_{ss})_i^n = \tau_0^{sur} S_\Gamma \mathbf{Y}^T(x_i^n) \mathbf{Y}(x_i^n)$$

where NIP is the

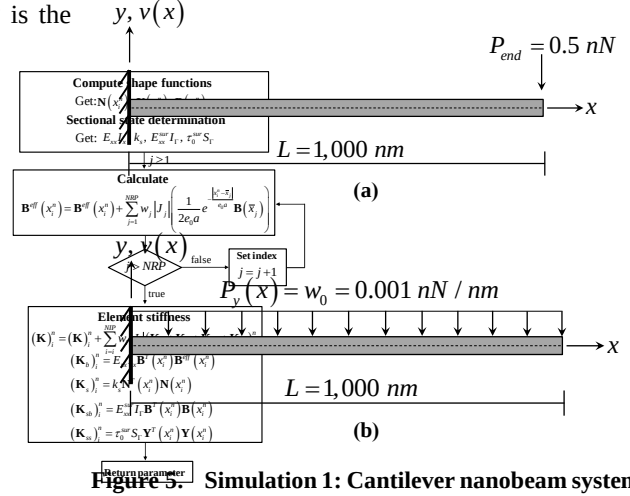


Figure 4. Iterative algorithm subjected to (a) free-end concentrated element stiffness load; and (b) uniformly distributed load

number of integration points.

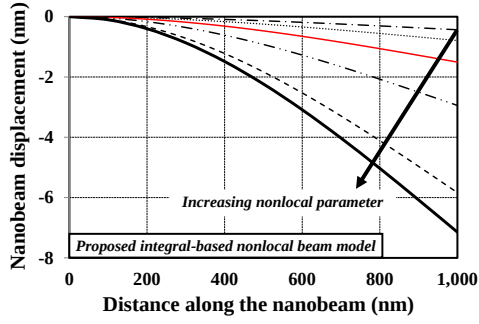
Numerical Simulations

In this section, three numerical simulations are employed to investigate the bending behavior of the proposed nanobeam model. The first simulation verifies the capability of the presented model to resolve the inconsistent responses of the Eringen nonlocal differential beam model (Limkatanyu *et al.*, 2018), while the second simulation addresses the impact of the surface-energy and nonlocality effects on the bending responses of the nanobeam problem on the substrate medium. Finally, the third simulation examines the system parameters for contact stiffness of the cantilever nanobeam system resting on the substrate medium.

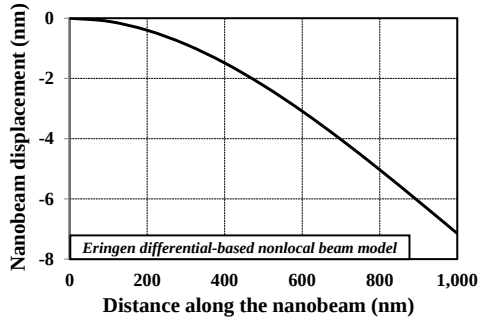
Simulation 1: Paradoxical resolve

As described by Norouzzadeh and Ansari (Norouzzadeh and Ansari, 2017) and several others (Peddieson *et al.*, 2003; Challamel *et al.*, 2014; Khodabakhshi and Reddy, 2015; Ceballes *et al.*, 2021; Sae-Long *et al.*, 2021), the paradoxical responses occur on using the Eringen nonlocal differential model in cases of the nanobeam problems when the cantilever beams have free-end concentrated load or uniformly distributed load. To demonstrate the capacity of the proposed model to remedy these inconsistent results, two cantilever circular silver nanobeam systems subjected to a concentrated load of 0.5 nN at the free-end, as shown in Figure 5(a), and subjected to the uniformly distributed load of 0.001 nN/nm, as shown in Figure 5(b), are used here. Only the nonlocal effect is of interest in this simulation. Therefore, the surface energy and substrate-structure interaction are not considered ($E_{xx}^{sur} = \tau_0^{sur} = k_s = 0$).

The material properties of a silver nanobeam are given by He and Lilley (He and Lilley, 2008) as: the beam bulk modulus $E_{xx} = 76 \text{ GPa}$ and the Poisson's ratio $\nu = 0.37$. The geometric properties of nanobeam are set to a constant span length of $L = 1,000 \text{ nm}$, and diameter of $d = 50 \text{ nm}$. In each nanobeam system, three beam models are used to compare the nanobeam responses, namely: (a) *proposed integral-based nonlocal beam model*; (b) *Eringen differential-based nonlocal beam model* (Limkatanyu *et al.*, 2018); and (c) *classical beam model*. The so-called “classical” model is obtained by not accounting for the nonlocal parameter (i.e., by setting $e_0 a = 0$). Furthermore, this simulation varies the normalized nonlocal parameter $\zeta = e_0 a / L$ within the range 0.025-0.4 to show the influence of the nonlocal parameter on the beam deflection. To approximate the analytical responses of the proposed model, 64 proposed elements with seven classical Gauss integration points per element and seven reference points per element are employed. These suffice as fine enough discretization to represent the limiting response of the complicated integral model of nonlocal elasticity.



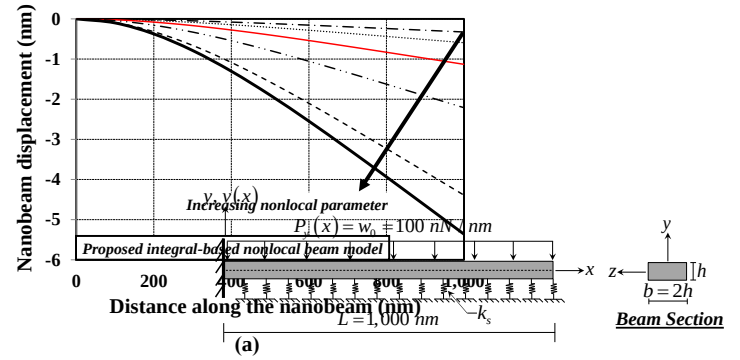
(a)



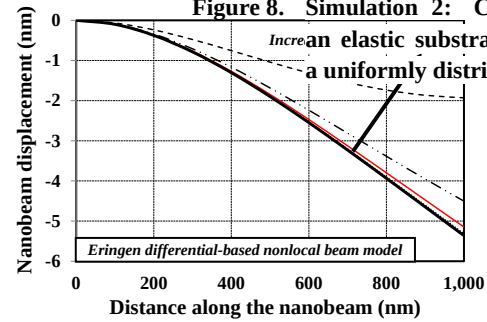
(b)

Figure 6. Transverse displacement versus distance along the nanobeam for Example I (a): (a) proposed beam model; and (b) Eringen differential-based nonlocal beam model

Figures 6(a) and 6(b) present the beam deflection of the cantilever nanobeam system subjected to the concentrated load, as illustrated in Figure 5(a). The diagram in Figure 6(a) reveals that the proposed model can capture the influence of the small-scale effect on the beam response by softening beam stiffness as the nonlocal parameter increases. On the other hand, the response as obtained from the beam model based on the differential form of the nonlocal elasticity and that obtained from the classical model are identical even if the nonlocal parameter increases (a paradoxical phenomenon) as shown in Figure 6(b). This observation implies that the small-scale effect corresponding to the Eringen nonlocal differential beam model is independent on the beam deflection in this particular case (Peddieson *et al.*, 2003).



(a)



(b)

Figure 7. Transverse displacement versus distance along the nanobeam for Example I

Figures 7(a) and 7(b) illustrate the transverse displacement of the cantilever nanobeam system under the uniformly distributed load shown in Figure 5(b). As similarly observed in Figure 6(a), the increase of nonlocality leads to softening of the element stiffness for the proposed model, as shown in Figure 7(a). In contrast, the beam stiffness increases following an increase of the nonlocal parameter for the differential-based nonlocal beam model, as seen in Figure 7(b). Challamel *et al.* (Challamel *et al.*, 2014) pointed out this inconsistency of the Eringen differential model, as it is not self-adjoint, which is related to having non-conservative moment inertia at the free-end boundary and so the model cannot be based on a potential (Challamel *et al.*, 2014; Reddy and El-Borgi, 2014).

According to the results from these simulations, the capability of the proposed model to represent nanobeam behavior and to resolve the paradoxical responses from cantilever nanobeam systems have been demonstrated. Furthermore, the nonlocality makes the beam

stiffness soften. This observation matches well the work of Oskouie *et al.* (Oskouie *et al.*, 2018).

Simulation 2: Size-scale effects in the bending analysis of nanobeam on elastic substrate

To examine the implications of the small-scale (nonlocal) and size-dependent (surface-energy) effects within the nanobeam model on substrate medium, the cantilever rectangular nanobeam on elastic substrate medium under a uniformly distributed load, as shown in Figure 8, is employed in this simulation. The amplitude of the uniformly distributed load was set at $w_0 = 100 \text{ nN/nm}$, while the beam cross-section was $200 \text{ nm} \times 100 \text{ nm}$ with a constant length of $1,000 \text{ nm}$. The material properties of the iron nanobeam follow those used by Mahmoud *et al.* (Mahmoud *et al.*, 2012) and Lim and He (Lim and He, 2014). The beam bulk modulus E_{xx} and Poisson's ratio ν of the iron nanobeam are 56.25 GPa and 0.25 , while the surface-layer modulus E_{xx}^{sur} and its residual surface stress τ_0^{sur} are $23,000 \text{ nN/nm}$ and 110 nN/nm , respectively. Regarding the substrate medium properties, this simulation uses the polymer substrate medium with a stiffness coefficient K_s of $95 \times 10^{-3} \text{ nN/nm}^3$ as given by Liew *et al.* (Liew *et al.*, 2006), the elastic substrate stiffness k_s can be defined as 19 nN/nm^2 , and the nonlocal parameter ($e_0 a$) is kept constant at 200 nm as employed by Yang and Lim (Yang and Lim, 2012). To demonstrate the significance of the nonlocal and surface-energy effects on the bending analysis, four beam models are employed to compare their bending responses, namely: (a) *proposed model with surface effect*; (b) *proposed model without surface effect*; (c) *classical model with surface effect*; and (d) *classical model*.

Based on the geometric properties of the nanobeam, the sectional properties of the proposed model are:

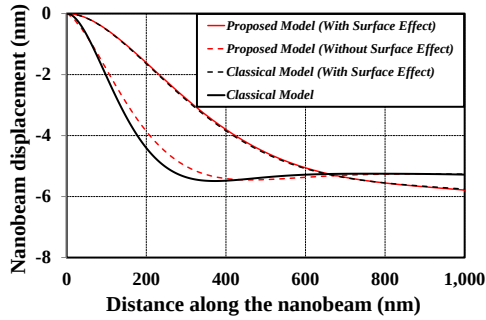
$$I_x = \frac{bh^3}{12}; I_T = \frac{h^3}{6} + \frac{bh^2}{2}; \text{ and } S_T = 2b \quad (41)$$

Figure 9(a) shows the transverse displacement profiles obtained from the four alternative models. Based on the set of parameters, this diagram reveals that both the nonlocal and the surface-energy effects have obviously an influence on the beam deflection, especially the surface-energy effect. This is based on the fact that the surface effect induces the elements to be stiffer, via its contribution in Equation (40) for stiffness. The term of the surface flexural rigidity $E_{xx}^{sur} I_T$ improves the system

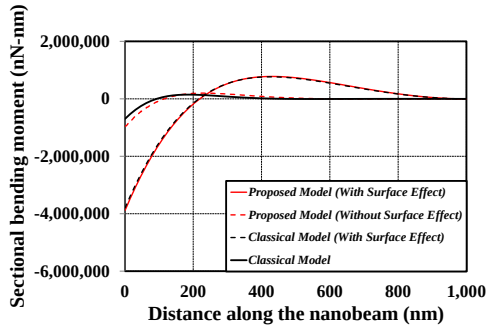
flexural stiffness (increases it) as demonstrated in Equation (33). Furthermore, it is interesting to point out that the term of residual surface stress τ_0^{sur} in the governing differential Equation (33) is similar to the term of the shear-layer component for the Winkler-Pasternak foundation model (Limkatanyu *et al.*, 2015b). This can imply that the residual surface stress is equivalent to the shear resisting component, which contributes the Winkler-springs to be interactive, resulting in increased element stiffness.

Figure 9(b) presents the bending moment diagram along the nanobeam length obtained from the four alternative models. Due to its restrain condition, the maximum moment occurs at the clamped support. This position can indicate that the nonlocality and surface-energy effects lead to an increased maximum moment (Negative), especially the surface effect when compared to the classical model. Similarly, the positive bending moment caused by the influence of the substrate medium tends to depend on both of the size-scale effects.

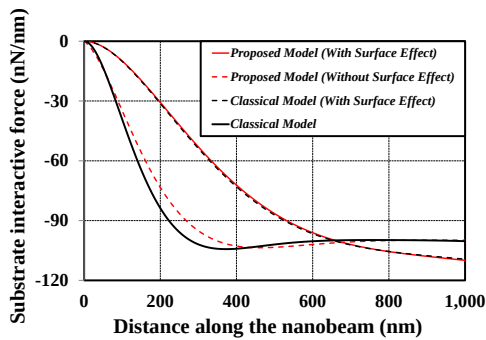
Figure 9(c) shows the distribution of the substrate interactive forces $D_s(x)$ along the nanobeam length as derived from the four alternative models. Based on Winkler's hypothesis (Limkatanyu *et al.*, 2013), the substrate interactive force is obtained from the transverse displacement $v(x)$ through the substrate constitutive relation in Equation (13). As a result, the shape of this diagram is similar to the deflection diagram in Figure 9(a).



(a)



(b)



(c)

Figure 9. Diagram of the responses versus distance along the nanobeam for simulation II: (a) transverse displacement; (b) bending moment; and (c) substrate interactive force

Furthermore, it is observed for this diagram that the surface-energy and nonlocal effects lead to reduction of the substrate interactive forces in the leftward portion at approximately halfway in the nanobeam-elastic substrate system. Conversely, the residue portion in the rightward side is slightly increased by both size-scale effects. This particular behavior is caused by the restraint condition of the indeterminate nanobeam system and the substrate-structure interactions.

Simulation 3: Parametric studies

The last simulation investigates the implications of the system parameters on the contact stiffness K_c . The cantilever circular silver nanobeam system on the elastic substrate medium subjected to an end-point load of 10 nN, as shown in Figure 10, is employed here. The material properties of the silver nanobeam and its surface follow those employed by He and Lilley (He and Lilley, 2008) as follows: the beam bulk modulus E_{xx} is 76 GPa; Poisson ratio ν is 0.37; surface elastic modulus E_{xx}^{sur} is 1.22 nN/nm; and residual surface stress τ_0^{sur} is 0.89 nN/nm. In this simulation, the so-called “classical model”, which is the conventional beam model on the elastic foundation (Limkatanyu *et al.*, 2013) without considering either the surface-energy or the nonlocal effects, is employed for comparison. To investigate the sensibility of the system parameters on the contact stiffness, the slenderness ratio L/d , normalized nonlocal-scale parameter $\zeta = e_0 a / L$, and non-dimensional substrate modulus $\bar{K}_s$ are selected and varied in the ranges $L/d = 10-50$, $\zeta = 0-0.2$, and $\bar{K}_s = 1-100$, while the span length L is held constant at 1,000 nm.

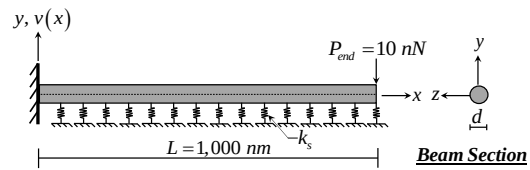


Figure 10. Simulation 3: Parametric studies

Based on the sectional shape, the sectional properties of circular nanobeam can be determined:

$$I_x = \frac{\pi d^4}{64}; \quad I_T = \frac{\pi d^3}{8}; \quad \text{and} \quad S_T = 2d \quad (42)$$

As suggested by Sea-Long *et al.* (Sea-Long *et al.*, 2021), the non-dimensional substrate modulus $\bar{K}_s$ and the contact stiffness K_c and $K_c^{classical}$ are simply based on the research of Khajeansari *et al.* (Khajeansari *et al.*, 2012) and Jiang and Yan (Jiang and Yan, 2010) as shown in Equation (43).

$$\bar{K}_s = \frac{k_s L}{(E_{xx} I_x + E_{xx}^{sur} I_\Gamma)}; K_c = \frac{P_{end}}{v_{end}}; \quad (43)$$

$$\text{and } K_c^{classical} = \frac{P_{end}}{v_{end}^{classical}}$$

where $\bar{K}_s$ is the non-dimensional substrate modulus; K_c and $K_c^{classical}$ are the contact stiffnesses obtained from the proposed model and the classical model, respectively; P_{end} is the imposed end-point load at free-end of the nanobeam; and v_{end} and $v_{end}^{classical}$ are the transverse displacements at the free-end of the nanobeam obtained from the proposed model and from the classical model, respectively.

Figure 11 plots the variation of the normalized contact stiffness $K_c / K_c^{classical}$ with the various values of the slenderness ratio L/d , normalized nonlocal- scale parameter $\zeta = e_0 a / L$, and non-dimensional substrate modulus $\bar{K}_s$. This diagram exposes that both the nonlocality and surface-energy effects influence contact stiffness, especially the surface effect. It can be observed that the normalized contact stiffness gradually increases with slenderness ratio (decreasing the nanobeam diameter). This result is based on the fact that the surface-energy effect is more dominant when the beam surface area/section area ratio (A_s / A_B) increases, in association with increase of the slenderness ratio, as concluded by Ponbunyanon *et al.* (Ponbunyanon *et al.*, 2016). Regarding influence of the nonlocality, this effect leads to increases normalized contact stiffness. However, the influence of both nonlocality and surface effects on the contact stiffness decreases as the non-dimensional substrate modulus is increased (for stiffer substrate media).

As observed in the second and third simulations, the nonlocality, surface-energy, and substrate- structure interaction effects play an essential role in the bending responses of the nanobeam system resting on the substrate medium.

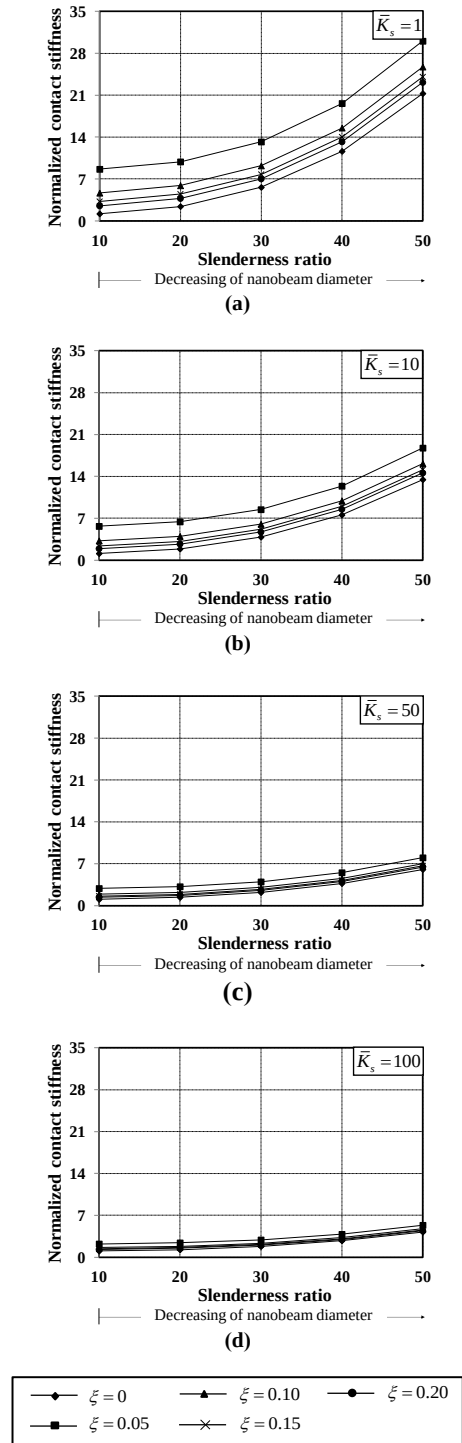


Figure 11. Normalized contact stiffness with varied slenderness ratio and non-dimensional substrate modulus: (a) $\bar{K}_s = 1$; (b) $\bar{K}_s = 10$; (c) $\bar{K}_s = 50$; and (d) $\bar{K}_s = 100$

Conclusions

The current paper presents a novel nanobeam-elastic substrate model for the bending analysis of an Euler-Bernoulli nanobeam resting on an elastic substrate medium. The Eringen nonlocal integral model is employed to address the beam-bulk nonlocality within the proposed model, while the surface elasticity theory and the Winkler-foundation model are used to represent the surface-energy and substrate-structure interaction effects, respectively. The proposed model is formulated with displacement-based finite element approach. Thus, the governing differential equation and its associated natural boundary conditions are obtained from the principle of virtual displacement. To evaluate the complex integral term, an iterative procedure and its deep details are proposed here. Finally, three numerical simulations were presented to investigate the capability of the proposed model to represent the bending behavior of a nanobeam system. The conclusions drawn are as follows:

The first simulation proved competence of the proposed model to resolve the inconsistent responses reported for Eringen nonlocal differential model applied to cantilever nanobeam problems with free-end point load or uniformly distributed load. An increase in the nonlocal parameter leads to softening of response in this particular case. This softening matches theoretical results proposed in prior literature (Oskouie *et al.*, 2018).

The second simulation demonstrated the impacts of nonlocality and surface-energy on the bending responses of a cantilever nanobeam system on an elastic substrate medium. From the results, it can be concluded that the surface-energy and nonlocal effects increased stiffness of the nanobeam, especially the surface effect. From the influences of the surface energy and its residual, the deflection and substrate interactive force are reduced while the maximum moment is increased.

The last simulation investigated the system parameters of the proposed nanobeam-elastic substrate model for relation to contact stiffness. The results show that the surface-energy and nonlocality effects increased the contract stiffness. The surface-energy effect is more pronounced when the beam becomes more slender. This stiffening phenomenon is caused by increased ratio of the beam surface area/section area, corresponding to increased slenderness. However, both size-scale effects decrease when the substrate medium becomes stiffer. Therefore, the nonlocality, surface-energy, and substrate-structure interaction effects are essential in the bending analysis of nanobeam on substrate problems.

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