

GROUP ANALYSIS OF TWO-TEMPERATURE RELAXATION GAS DYNAMICS EQUATIONS WITH CYLINDRICAL AND SPHERICAL SYMMETRIES

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Abstract

The two temperature gas dynamics equations with the Landau- Teller equation are considered in the paper. For solutions with spherical and cylindrical symmetry the gas dynamics equations reduce to a system of three partial differential equations. The group analysis method is applied to the study of these equations. As the result we found an admitted Lie group, constructed an optimal system of subalgebras. Using the optimal system of subalgebras all representations of nonequivalent solutions are obtained. Substituting the representations into gas dynamics equations with the Landau-Teller equation we derived the reduced equations. Analysis of the reduced equations is also given in the paper.

Keywords: Landau-Teller equation, admitted Lie group, cylindrical and spherical symmetries, invariant solutions

Introduction

Phenomena behind strong shock waves include excitation of vibrational modes of the molecules, dissociation, ionization, radiation heat transfer, catalytic activity of the wall and other processes. In these flow regions, the translational, rotational and vibrational temperatures of molecules are different, i.e., there is no thermal equilibrium. Under thermal non-equilibrium conditions, the dissociation

is principally different from that under equilibrium conditions: its rate depends on the translational and vibrational temperatures of the dissociating molecules (Brun, 2009). The relaxation region size can be comparable with the characteristic scale of the problem being solved; therefore, the flow structure is significantly affected by non- equilibrium phenomena. To describe behavior of a gas in the relaxation

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region the system of equations of two-temperature relaxation gas dynamics model is used. In this model vibrational relaxation is described by the inhomogeneous Landau-Teller equation. The equations describing the mass, momentum and energy balances are (Grigoryev and Ershov, 2017).

$$\frac{d\rho}{dt} + \rho \operatorname{div} v = 0, \quad (1)$$

$$\rho \frac{dv}{dt} + \nabla p = 0, \quad (2)$$

$$\frac{dT}{dt} + (\gamma - 1)T \operatorname{div}(v) + g \left(\frac{T_v}{T} \right) \frac{T - T_v}{\tau} = 0, \quad (3)$$

$$\frac{dT_v}{dt} = g \left(\frac{T_v}{T} \right) \frac{T - T_v}{\tau}, \quad (4)$$

where t is time, u is velocity, ρ is density, T is the translational-rotation temperature and T_v is the vibrational temperature, the thermodynamic pressure is given by the state equation

$$p = \rho R_0 T,$$

where R_0 is the gas constant. Equation (4) is the Landau-Teller equation. Here the operator $\frac{d}{dt}$ is the operator of the material derivative.

Scientists have always been interested in symmetry. The group analysis method (Ovsianikov, 1982); Olver, 1986; Ibragimov, 1999; Meleshko, 2005) is one of the tools for studying symmetry. It is the basic method for creating exact solutions of partial differential equations.

Using analysis of admitted Lie group, invariant and partially invariant solutions can be constructed. The objective research of the present paper is to apply the group analysis method to the gas dynamics equations with the Landau-Teller equation, where the solutions have either spherical or cylindrical symmetry.

The paper is organized as follows. Section 2 the studied equations are presented.

In the next section, the admitted Lie group of the studied equations is analyzed. The invariant solutions are considered in Section 4. Some of numerical solutions of the reduced equations are given in Section 5. Conclusions are presented at the end of the paper.

Equations in Cylindrical and Spherical Coordinates

Equations (1) for solutions with spherical or cylindrical symmetry are reduced to the equations.

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \left(\frac{\partial u}{\partial x} + \frac{nu}{x} \right) = 0, \quad (5)$$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} = 0, \quad (6)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + (\gamma - 1)T \left(\frac{\partial u}{\partial x} + \frac{nu}{x} \right) = -\gamma_v \left(\frac{T - T_v}{\tau} \right), \quad (7)$$

$$\frac{\partial T_v}{\partial t} + u \frac{\partial T_v}{\partial x} = \frac{T - T_v}{\tau}, \quad (8)$$

where $n = 1$ for cylindrical symmetry, and $n = 2$ for spherical symmetry.

Notice that using the dimensionless analysis, one can assume that $R_0 = 1$ and $\tau = 1$.

Admitted Lie Group of Equations (5)-(8)

The calculations show that Equations (5)-(8) admit the Lie group corresponding to the Lie algebra L_3 with the basis generators

$$X_1 = \partial_t, \quad X_2 = 2T_v \partial_{T_v} + 2T \partial_T + u \partial_u + x \partial_x, \quad X_3 = \rho \partial_\rho,$$

The admitted Lie algebra is Abelian. Hence, the optimal system of one-dimensional subalgebras consists of the subalgebras

$$\{X_1 + \beta X_3\}, \quad \{X_2 + \alpha X_1 + \beta X_3\}, \quad \{X_3\}.$$

Invariant Solutions

For finding essentially different invariant solutions one needs to choose a subalgebra from the optimal system of subalgebras. For finding invariants one should to find a complete set of functionally independent solutions of the equation $XI = 0$, where the generator X is taken from the optimal system of subalgebras.

Obtaining invariants of the chosen Lie algebra, we derive a representation of the invariant solution. Substituting the representation of the invariant solution into Equations (5)-(8), we derive equations with reduced number of the independent variables. These equations are called reduced equations.

The Subalgebra $\{X_1 + \beta X_3\}$

The invariants of the generator $X_1 + \beta X_3$ are $x, \rho e^{\beta t}, u, T, T_v$.

The presentation of the invariant solutions is $\rho = e^{\beta t} R(x)$, $u = U(x)$, $T = T(x)$, $T_v = T_v(x)$.

Substitution of this representation into Equations (5)-(8) leads to the reduced equations

$$\frac{\partial(RU)}{\partial x} + \frac{nRU}{x} = \beta R, \quad (9)$$

$$\frac{\partial(RT)}{\partial x} + UR \frac{\partial U}{\partial x} = 0, \quad (10)$$

$$U \frac{\partial T}{\partial x} + (\gamma - 1)T \frac{\partial U}{\partial x} + \gamma_v(T - T_v) + nUT \frac{(\gamma - 1)}{x} = 0, \quad (11)$$

$$\frac{\partial T_v}{\partial t} = T - T_v. \quad (12)$$

Notice that if $U = 0$, then $\beta = 0$ and $T_v = T = \frac{C_1}{R}$,

where $R(x)$ is an arbitrary function. In this case the gas is in a rest at thermodynamic equilibrium.

If $U \neq 0$, then from Equation (9), (11), (12) one finds

$$\begin{aligned} \frac{\partial U}{\partial x} &= -\frac{U}{R} \frac{\partial R}{\partial x} + \beta - \frac{nU}{x}, \\ \frac{\partial T}{\partial x} &= \frac{U^2 - T}{R} \frac{\partial R}{\partial x} + U \left(\frac{nU}{x} - \beta \right), \\ \frac{\partial T_v}{\partial x} &= \frac{T - T_v}{U} \end{aligned} \quad (13)$$

Equation (10) becomes

$$Ux(U^2 - \gamma T) \frac{\partial R}{\partial x} + R(\beta T x(\gamma - 1) - \beta U x + \gamma_v x(T - T_v) + nU^3) = 0 \quad (14)$$

Further analysis depends on the coefficient related with $\frac{\partial R}{\partial x}$ in the latter equation.

Consider the case where $(U^2 - \gamma T) = 0$. The first and second Equations of (13) become

$$\frac{\partial U}{\partial x} = \frac{nU - \beta x}{x(\gamma + 1)}, \quad \frac{\partial R}{\partial x} = \frac{R((\gamma + 2)(\beta x - nU))}{Ux(\gamma + 1)}.$$

Equation (14) reduces to

$$T_v \gamma \gamma_v x = U^2(-\beta x + \gamma nU + \gamma_v x). \quad (15)$$

If $\gamma_v = 0$, then $U = \frac{\beta x}{\gamma n}$ and it gives the solution of the classical gas dynamics equations, where, R, U are constant and $T = \frac{\beta^2 x^2}{\gamma^3 n^2}$. The function T_v satisfies a simple linear second-order ordinary differential equation (the third Equation of (13)).

If $\gamma_v \neq 0$, then

$$T_v = \frac{U^2(-\beta x + \gamma nU + \gamma_v x)}{\gamma \gamma_v x}. \quad (16)$$

Substituting T_v into the third equation of (13), one derives that

$$\begin{aligned} &U^2 \gamma n(\gamma - 3n + 1) \\ &+ nUx(3\beta \gamma + 2\beta - \gamma^2 - \gamma + 2\gamma_v) \\ &+ \beta x^2(-2\beta + \gamma + 2\gamma_v + 1) = 0 \end{aligned} \quad (17)$$

Consider the case where $\gamma = 3n - 1$. From Equation (17) one obtains that $\beta(9n - 1) - 2\gamma_v - 9n^2 + 3n \neq 0$, and $U = \frac{\beta x(2(\beta - \gamma_v) - 3n)}{n(\beta(9n - 1) - 2\gamma_v - 9n^2 + 3n)}$,

Substituting these relations into Equations (13), one finds that

$$\begin{aligned}\beta &= \frac{2\gamma_v + 3n^2 + n}{3n + 1}, \quad U = \frac{x(-2\gamma_v - 3n^2 - n)}{2n(3n + 1)}, \\ T &= \frac{x^2(2\gamma_v + 3n^2 + n)^2}{4n^2(3n + 1)(9n^2 - 1)}, \\ T_v &= -\frac{x^2(2\gamma_v + 3n^2 + n)^2}{8n\gamma_v(9n^2 - 1)}, \\ R &= Rx^{-3n},\end{aligned}$$

where R is constant.

Because $T_v < 0$, this solution has no physical meaning.

Consider the case where $\gamma \neq 3n - 1$. After similar calculations one obtains that

$$\begin{aligned}U &= \frac{x(-\gamma n - \gamma - 2\gamma_v + n - 1)}{2(\gamma n - \gamma - n + 1)}, \\ R &= R_0 x^{\rho(-\gamma-1)}, \\ T &= \frac{x^2(\gamma n + \gamma + 2\gamma_v - n + 1)^2}{4\gamma(\gamma n + \gamma - n + 1)^2}, \\ T_v &= -\frac{x^2(\gamma n + \gamma + 2\gamma_v - n + 1)^2}{8\gamma\gamma_v(\gamma n + \gamma - n + 1)}.\end{aligned}$$

Notice that in this case also $T_v < 0$, which is not physical.

Case ($U^2 - \gamma T \neq 0$). In this case one finds that the function U , R , T and T_v satisfy a normal system of ordinary differential equations:

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{\beta T x - \gamma n T U - \gamma_v x(T + T_v)}{x(\gamma T - U^2)}, \\ \frac{\partial R}{\partial x} &= \frac{R(\beta T x(\gamma - 1) - \beta U^2 x + \gamma_v x(T - T_v) + n U^3)}{x U(\gamma T - U^2)}, \\ \frac{\partial T}{\partial x} &= \frac{\beta T^2 x(1 - \gamma) + \gamma n T U^3 - \gamma_v x(T^2 - T T_v) + \gamma_v U^2 x(T - T_v) - n T U^3}{U x(\gamma T - U^2)}, \\ \frac{\partial T_v}{\partial x} &= \frac{T - T_v}{U}.\end{aligned}\quad (18)$$

The Subalgebra $\{X_2 + \alpha X_1 + \beta X_3\}_{\alpha=0}$

As invariants of the generator $X_2 + \beta X_3$ are t , $\rho x^{-\beta}$, $x^{-1}u$, $T x^{-2}$, $T_v x^{-2}$. a representation of the invariant solution is $\rho = x^\beta R(t)$, $u = x U(t)$, $T = x^2 \tilde{T}(t)$, $T_v = x^2 \tilde{T}_v(t)$.

Substitution of the representation of the invariant solution into (5)-(8) leads to the normal system of reduced equations

$$\begin{aligned}\frac{\partial U}{\partial t} &= -\tilde{T}(\beta + 2) - U^2, \\ \frac{\partial R}{\partial t} &= -R U(\beta + n + 1), \\ \frac{\partial \tilde{T}}{\partial t} &= -\gamma \tilde{T} U(n + 1) - \gamma_v(\tilde{T} - \tilde{T}_v) + \tilde{T} U(n - 1), \\ \frac{\partial \tilde{T}_v}{\partial t} &= \tilde{T} - \tilde{T}_v(2U + 1).\end{aligned}\quad (19)$$

The Subalgebra $\{X_2 + \alpha X_1 + \beta X_3\}_{\alpha \neq 0}$

Invariants of the generator are $x e^{t\alpha}$, $u e^{t\alpha}$, $\rho e^{-\beta t\alpha}$, $T e^{-2t\alpha}$, $T_v e^{-2t\alpha}$.

A representation of the invariant solution is $\rho = e^{\beta t\alpha} R(z)$, $u = e^{t\alpha} U(z)$, $T = e^{2t\alpha} \tilde{T}(z)$, $T_v = e^{2t\alpha} \tilde{T}_v(z)$,

where $z = x e^{t\alpha}$. Substituting the representation of the invariant solution into Equations (5)-(8), one has to separate the study into two cases.

Case 1 : $U = \alpha z$. One finds that

$$\beta = -(n + 1), \quad \alpha = \frac{-\gamma(n + 1) - 2\gamma_v + n - 1}{2\gamma(n + 1) + 2(-n + 1)},$$

$$U = \frac{-\gamma z(n + 1) - 2\gamma_v z + z(n - 1)}{2\gamma(n + 1) + 2(-n + 1)},$$

$$\tilde{T} = -\frac{2\gamma_v \tilde{T}_v}{\gamma(n + 1) - n + 1},$$

and the derivative $\frac{\partial R}{\partial z}$ is

$$n = 1: \quad \frac{\partial R}{\partial z} = \frac{-4\tilde{T}_v \gamma \gamma_v R + \gamma^2 R z + \gamma_v R z(2\gamma + \gamma_v)}{4\gamma \gamma_v \tilde{T}_v},$$

$n = 2:$

$$\frac{\partial R}{\partial z} = \frac{8\tilde{T}_v \gamma_v R(-3\gamma + 1) + \gamma R z(9\gamma + 12\gamma_v) - R z(6\gamma - 4\gamma_v^2 + 4\gamma_v - 1)}{8\gamma_v \tilde{T}_v(3\gamma - 1)}.$$

Notice also that as $T < 0$ in this case, this solution has no physical meaning.

Case 2 : $U \neq \alpha z$.

Let $(\alpha z - U)^2 = \gamma T$, then

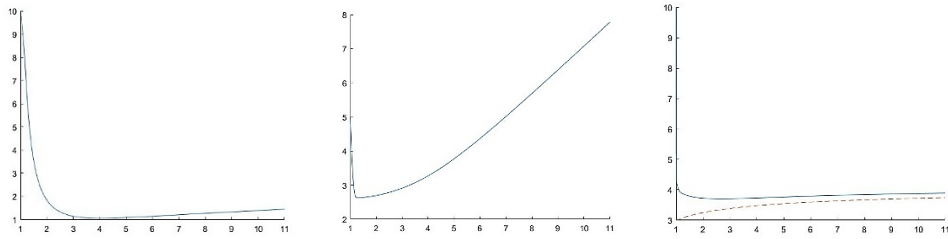


Figure 1. Numerical solution of Equations (18) with the parameters $\beta = 1$, $\gamma = 1.4$, $\gamma_v = 2$ and the initial data $\rho_0 = 10$, $v_1 = 5$, $T_0 = 10$ and $T_{v0} = 3$

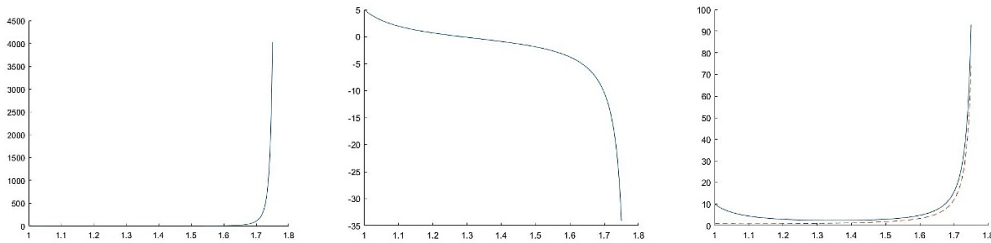


Figure 2. Numerical solution of Equations (19) with the parameters $\beta = 1$, $\gamma = 1.4$, $\gamma_v = 2$ and the initial data $\rho_0 = 10$, $v_1 = 5$, $T_0 = 10$ and $T_{v0} = 1$

$$R = \frac{R_0}{z^{2+(k^2\gamma)/(k^2-2k\alpha-\alpha^2)}}, \quad U = kz,$$

$$T = \frac{(k-\alpha)^2}{\gamma} z^2, \quad T_v = \frac{kz(1-2\alpha z) + \alpha^2 z^2}{\gamma(2k+1)},$$

$$\beta = \frac{k^2(-\gamma+n-1) + k\alpha(-n+3) - 2\alpha^2}{-k\alpha + \alpha^2},$$

and $k = -\frac{\gamma(n+1) - n + 1 + \gamma_v}{2(\lambda(n+1) - n + 1)}$, where R_0 is constant.

In case $(\alpha z - U)^2 \neq \gamma T$, all derivatives of the functions U , R , $\tilde{T}$, and $\tilde{T}_v$ can be found. Because of their cumbersome expressions they are not presented here.

Numerical analysis of solutions of Equation (18) and (19)

The sixth-order Runge-Kutta method was used for numerical study of solutions of Equations (18) and (19). The results of this study are presented in Figures 1 and 2. Figure 1 presents a solution invariant with respect to the

subalgebra $\{X_1 + \beta X_3\}$, which is described by Equations (18). Figure 2 presents a solution invariant with respect to the subalgebra $\{X_2 + \alpha X_1 + \beta X_3\}_{\alpha=0}$, which is a numerical solution of Equations (19). The first picture of the Figures relates with the density, the second picture presents the velocity and the third picture describes the temperatures. From the numerical results one can see that the static and vibrating temperatures tend to relaxation.

Conclusions

In the present paper exact solutions with spherical and cylindrical symmetry of two-temperature relaxation gas dynamics are presented. Some found invariant solutions are in explicit form and some invariant solutions are reduced to integration of a normal system of ordinary differential equations. Although the found solutions in explicit form have no physical meaning, they can be used for testing numerical schemes.

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