

ON GROUP CLASSIFICATION OF ONE-DIMENSIONAL EQUATIONS OF NONISENTROPIC FLUIDS WITH INTERNAL INERTIA

Piyanuch Siriwat*

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Abstract

One-dimensional flows of fluids with internal inertia are studied in the manuscript. The given equations include such models as the non-linear one-velocity model of a bubbly fluid (with an incompressible liquid phase) at a small volume concentration of gas bubbles and the dispersive shallow water model. These models are obtained for special types of the function $W(\rho, \dot{\rho}, S)$. Earlier the case of $W_{\dot{\rho}, \dot{\rho}, S} = 0$ was studied. Here the study of the case of $W_{\dot{\rho}, \dot{\rho}, S} \neq 0$ is started.

Keywords: Group classification, equivalence Lie group, admitted Lie group, fluids with internal inertia

Introduction

Symmetry is a fundamental topic in many areas of physics and mathematics (Olver, 1993; Marsden and Ratiu, 1994; Golubitsky and Stewart, 2002). Whereas group-theoretical methods play a prominent role in modern theoretical physics, a systematic use of them in constructing models of continuum mechanics has not been widely applied yet (Ovsianikov, 1994). A class of dispersive models is given by a system of equations (Gavrilyuk and Teshukov, 2001)

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div}(u) &= 0, \quad \rho \dot{u} + \nabla p = 0, \\ p &= \rho \frac{\delta W}{\delta \rho} - W = \rho \left(-\frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t} \left(\frac{\partial W}{\partial \dot{\rho}} \right) - \operatorname{div} \left(\frac{\partial W}{\partial \dot{\rho}} u \right) \right) - W, \end{aligned} \quad (1)$$

where t is time, ∇ is the gradient operator with

respect to the space variable, ρ is the fluid density, u is the velocity field, $W(\rho, \dot{\rho}, S)$ is a given potential, a superposed dot denotes the

material time derivative: $\dot{f} = \frac{df}{dt} = f_t + u \nabla f$, and $\frac{\delta W}{\delta \rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u . These models include the non-linear one-velocity model of a bubbly fluid (with an incompressible liquid phase) at a small volume concentration of gas bubbles (Iordanskii, 1960; Kogarko, 1961; Wijngaarden, 1968), and the dispersive shallow water model (Green and Naghdi, 1976; Salmon, 1998). Equations (1) were obtained in Gavrilyuk and Teshukov (2001), using the Lagrangian of the form

School of Science, Mae Fah Luang University, Chiangrai, 57100, Thailand. E-mail: fonluang@yahoo.com

* *Corresponding author*

$$L = \frac{1}{2} |u|^2 - W(\rho, \dot{\rho}, S).$$

The behavior of such a continuum along thermo dynamical variables also depends on the derivatives with respect to space and time. In this particular case the potential function depends on the total derivative of the density which reflects the dependence of the medium on its inertia.

The main tool for modeling in the present paper is the group analysis method (Ovsiannikov, 1978; Olver, 1993; Ibragimov, 1999). This method is a basic method for constructing exact solutions of partial differential equations. A wide range of the applications of group analysis to partial differential equations is collected in Ibragimov, (1994, 1995, 1996). Group analysis, besides facilitating the construction of exact solutions, provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This feature of group analysis is the fundamental basis for mathematical modeling in the present paper.

A complete group classification of Equations (1), where $W = W(\rho, \dot{\rho})$ has been performed in Siriawat and Meleshko (2011) (a one-dimensional case) and Siriawat and Meleshko (2008) (a three-dimensional case). Invariant solutions of some particular cases which are separated out by group classification are considered in Salmon (1998); Hematulin *et al.* (2007); Siriawat and Meleshko (2008). Notice that the thermodynamics requires the inclusion of the entropy into the study $W = W(\rho, \dot{\rho}, S)$. It is also worth noticing that the classical gas dynamics model corresponds to $W = W(\rho, S)$ (or $\varepsilon = \varepsilon(\rho, S)$).

A complete group classification of the gas dynamics equations was presented in Ibragimov (1995). Later, an exhaustive program of the study of the models appearing in the group classification of the gas dynamics equations was announced in Ovsiannikov (1978). Some results of this program were summarized in

Ovsiannikov (1999). The analysis of Equations (1) with $W = W(\rho, \dot{\rho}, S)$ is more complicated than the analysis of the gas dynamics equations. Recently the group classification of the one-dimensional equations of fluids (1), where the function $W = W(\rho, \dot{\rho}, S)$ is satisfying the conditions $W_{\dot{\rho}\dot{\rho}S} = 0$ and $W_S \neq 0$, was studied (Siriawat and Meleshko, 2011). The present paper starts the study of the group classification of Equations (1), where $W_{\dot{\rho}\dot{\rho}S} \neq 0$.

Equivalence Lie Group

The group classification separates all Equations (1) into equivalent classes with respect to the equivalence of each Lie group. For the Equations (1) with $W = W(\rho, \dot{\rho}, S)$ this group was found in Siriawat and Meleshko (2011). The equivalence transformations consist of the mappings:

$$\begin{aligned} X_1^e: \quad \rho' &= \rho e^a, \quad \dot{\rho}' = \dot{\rho}, \quad W' = W e^{-a}; \\ X_2^e: \quad \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad W' = W; \\ X_3^e: \quad \rho' &= \rho e^{2a}, \quad \dot{\rho}' = \dot{\rho}, \quad W' = W; \\ X_6^e: \quad \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad W' = W e^{-2a}; \\ X_7^e: \quad \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad W' = W + a; \\ X_8^e: \quad \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad W' = \rho \varphi(S) a + W; \\ X_9^e: \quad \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad W' = \dot{\rho} g(\rho, S) a + W. \end{aligned}$$

Here a is the group parameter and, because the function W depends on ρ and $\dot{\rho}$ only the transformations of these variables are presented.

Admitted Lie Group

The determining equations of an admitted Lie group were obtained in Siriawat and Meleshko, (2011) and consist of the equations

$$\begin{aligned} W_{S\dot{\rho}\dot{\rho}} \dot{\rho} \xi &= W_{\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho}^2 k_1 + 2(W_{\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho} + W_{\dot{\rho}\dot{\rho}}) \\ \dot{\rho} k_3 - (W_{\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho} + W_{\dot{\rho}\dot{\rho}\dot{\rho}} \rho + W_{\dot{\rho}\dot{\rho}}) \dot{\rho} k_8 &= 0 \end{aligned} \quad (2)$$

$$\begin{aligned} (W_{\dot{\rho}\dot{\rho}\dot{\rho}} - W_{S\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho}) \rho \xi &+ (-W_{\dot{\rho}\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho} - W_{\dot{\rho}\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho}^2 - W_{\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho} \\ &+ W_{\dot{\rho}\dot{\rho}\dot{\rho}} \rho + W_{\dot{\rho}\dot{\rho}}) \rho k_8 + (W_{\dot{\rho}\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho}^2 - 2W_{\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho} + 2W_{\dot{\rho}\dot{\rho}}) \\ \rho k_1 &+ 2(W_{\dot{\rho}\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho}^2 - W_{\dot{\rho}\dot{\rho}\dot{\rho}} \dot{\rho} + W_{\dot{\rho}\dot{\rho}}) \rho k_3 = 0 \end{aligned} \quad (3)$$

$$\begin{aligned}
 & (-W_{s\rho\rho}\dot{\rho}p + W_{s\rho}\dot{\rho} + W_{s\rho}\rho - W_s)\zeta' \\
 & + (W_{ss\rho}\dot{\rho} - W_{ss\rho\rho}\dot{\rho}p + W_{ss\rho}\rho - W_{ss})\zeta \\
 & + (W_{s\rho\rho\rho}\dot{\rho}^2p - 2W_{s\rho\rho}\dot{\rho}p - W_{s\rho\rho}\dot{\rho}^2 + 2W_{s\rho}\dot{\rho} \\
 & + 2W_{s\rho}\rho - 2W_s)k_1 + 2(W_{s\rho\rho\rho}\dot{\rho}^2p - W_{s\rho\rho}\dot{\rho}p - W_{s\rho\rho}\dot{\rho}^2 - \\
 & + W_{s\rho}\dot{\rho} + W_{s\rho}\rho - W_s)k_3 + (W_{s\rho\rho}\dot{\rho}p - W_{s\rho\rho\rho}\dot{\rho}^2p + W_{s\rho\rho}\dot{\rho}^2 - \\
 & W_{s\rho}\dot{\rho} + W_{s\rho\rho}\rho^2 - W_{s\rho}\rho + W_s - W_{s\rho\rho\rho}\dot{\rho}p^2)k_8 = 0 \quad (4)
 \end{aligned}$$

where k_i , ($i = 1, 3, 4, 5, 7, 8$) are constant. The admitted generators have the form

$$X = k_4 Y_4 + k_5 Y_5 + k_6 Y_6 + k_1 X_1 + k_3 X_3 + k_8 X_8 + \zeta(S)\partial_S,$$

where

$$\begin{aligned}
 Y_4 &= t\partial_x + \partial_u, Y_5 = \partial_x, Y_6 = \partial_t, \\
 X_1 &= t\partial_t - u\partial_u - \rho\partial_\rho, X_3 = x\partial_x + 2t\partial_t - u\partial_u - \\
 &\rho\partial_\rho - 3\dot{\rho}\partial_\rho, X_8 = \rho\partial_\rho + \dot{\rho}\partial_\rho.
 \end{aligned}$$

Here there is excluded from the study of these cases the function $W(\rho, \dot{\rho}, S)$ such that Equations (1) admit projective transformations corresponding to the generator

$$X_2 = tx\partial_x + t^2\partial_t + (x-ut)\partial_u - t\rho\partial_\rho - (\rho+3t\dot{\rho})\partial_\rho.$$

A complete study of these cases was done in Siriwat and Meleshko (2008). The kernel of the admitted Lie algebras is defined by the generators Y_4, Y_5, Y_6 . The goal of the group classification is to find extensions of the kernel. The extensions depend on the value of the function $W(\rho, \dot{\rho}, S)$ They can only be operators of the form

$$k_1 X_1 + k_3 X_3 + k_8 X_8 + \zeta \partial_S.$$

Relations between the constants k_1, k_3 and k_8 depend on the function $W(\rho, \dot{\rho}, S)$.

The paper by Siriwat and Meleshko (2008) gives a complete solution to the group classification problem for the case where $W_{\dot{\rho}\dot{\rho}S} = 0$. The present paper deals with the function $W(\rho, \dot{\rho}, S)$ satisfying the condition $W_{\dot{\rho}\dot{\rho}S} \neq 0$. This assumption allows the finding of the coefficient ζ from Equation (2) and complicates further study comparing with Siriwat and Meleshko (2008). In the present

paper a complete solution of the group classification problem is presented for the cases where (a) $g_{\rho S} = 0$ and (b) $g_{\rho S} \neq 0$. Here $W_{\dot{\rho}\dot{\rho}} = e^g$. The study of the case where $g_{\rho S} = 0$ and $g_{\rho S} \neq 0$ will be presented in a future work.

The Study of Equations (1) with $W_{\dot{\rho}\dot{\rho}S} \neq 0$

Using the function $g(\rho, \dot{\rho}, S)$, Equation (2) is rewritten in the form

$$\tilde{k}_3 = \zeta g_S + k_8 \rho g_\rho - \tilde{k}_1 \dot{\rho} g_\rho, \quad (5)$$

Differentiating (5) with respect to ρ and one has

$$\zeta g_{\rho S} + k_8 (\rho g_\rho)_\rho - \tilde{k}_1 \dot{\rho} g_{\rho\rho} = 0 \quad (6)$$

$$\zeta g_{\rho S} + k_8 \rho g_{\rho\rho} - \tilde{k}_1 (\dot{\rho} g_{\rho\rho} + g_\rho) = 0 \quad (7)$$

$$\zeta g_{SS} + g_S \zeta' + k_8 \rho g_{\rho S} - \tilde{k}_1 \dot{\rho} g_{\rho S} = 0 \quad (8)$$

Case $g_{S\rho}$ and $g_{S\dot{\rho}} = 0$

Assume that $g_{S\rho} = 0$ and $g_{S\dot{\rho}} = 0$, and hence, $g = \varphi(S) + \tilde{\psi}(\rho, \dot{\rho})$ where $\varphi' \neq 0$. In this case the function

$$W(\rho, \dot{\rho}, S) = \mu(S)\psi(\rho, \dot{\rho}) + h(\rho, S),$$

where $\mu(S) = e^{\varphi(S)}$. Solving Equation (8), one finds that

$$\zeta = k \frac{\mu}{\mu'},$$

where k is constant. For an arbitrary function $\psi(\rho, \dot{\rho})$, Equations (6) and (7) lead to

$$k_8 = 0, \tilde{k}_1 = 0.$$

In this case Equation (5) becomes

$$\tilde{k}_3 = k, \quad (9)$$

and hence, for the existence of the extension of the kernel of admitted Lie algebras, one has $k \neq 0$. Because of the arbitrariness of the function $\psi(\rho, \dot{\rho})$ Equation (3) gives that the function h does not depend on ρ . Substituting W into Equation (4), up to the equivalence transformation, one obtains

$$h = 0.$$

Thus, the function

$$W(\rho, \dot{\rho}, S) = \mu(S)\psi(\rho, \dot{\rho}),$$

and the extension of the kernel is given by the generator

$$(-X_1 + X_3 / 2 + \tilde{S}\partial_{\tilde{S}}). \\ \tilde{X}_3 + \tilde{S}\partial_{\tilde{S}}.$$

Case $g_{sp} = 0$ and $g_{s\dot{p}} \neq 0$

Supposing that $g_{\rho\dot{\rho}} \neq 0$, from (6) and (7) one finds

$$\tilde{k}_1 = k_8 \frac{(\rho g_{\rho})_{\rho}}{\dot{\rho} g_{\rho\rho}}, \zeta = \left[\frac{(\dot{\rho} g_{\rho})_{\rho}}{g_{s\dot{\rho}}} \frac{(\rho g_{\rho})_{\rho}}{\dot{\rho} g_{\rho\rho}} - \frac{(\rho g_{\rho})_{\rho}}{g_{s\dot{\rho}}} \right] k_8. \quad (10)$$

Because, for $k_8 = 0$ there is no extension of the kernel, then $k_8 \neq 0$, and hence,

$$\frac{(\rho g_{\rho})_{\rho}}{\dot{\rho} g_{\rho\rho}} = \beta,$$

where β is constant. Integrating the last relation with respect to ρ , one has

$$\rho g_{\rho} - \beta \dot{\rho} g_{\rho} = h(\dot{\rho}, S) \quad (11)$$

Substituting ρg_{ρ} into the second equation of (10), and differentiating it with respect to $\dot{\rho}$, one finds that there exist functions $\alpha(S)$ and $\lambda(S)$ such that

$$h = \alpha g_s + \lambda$$

In this case Equations (10) and (5) become

$$\tilde{k}_1 = k_8 \beta, \zeta = -\alpha k_8, \tilde{k}_3 = k_8 \lambda.$$

From the last relation one derives that λ is constant. Hence, Equation (11) becomes

$$\rho g_{\rho} - \beta \dot{\rho} g_{\rho} - \alpha g_s = \lambda \quad (12)$$

Notice that because $g_{\rho\dot{\rho}} g_{\rho S} \neq 0$, the function $\alpha(S) \neq 0$. The general solution of Equation (12) is

$$g(\rho, \dot{\rho}, S) = \lambda \ln |\rho| + \tilde{\varphi}(\rho^{\beta} \dot{\rho}, \rho \mu(S)), \quad (13)$$

where $\alpha(S) = \mu(S) / \dot{\mu}(S)$. In this case the function

$$W(\rho, \dot{\rho}, S) = \rho^{\gamma} \varphi(\dot{\rho} \rho^{\beta}, \rho \mu(S)) + h(\rho, S), \quad (14)$$

where $\gamma = \lambda - 2\beta$. Substituting $W(\rho, \dot{\rho}, S)$ into Equation (3), we obtain

$$\mu h_{\rho\rho S} - \mu' \rho h_{\rho\rho\rho} = (2 - \gamma) \mu h_{\rho\rho} \quad (15)$$

Solving (15), one has

$$h(\rho, S) = \rho^{\gamma} \psi(\rho \mu) + h_0(S).$$

Changing the function φ (if necessary), one can assume that

$$h = h_0(S). \quad (16)$$

Substituting (16) into (4), one obtains the conditions

$$\frac{h_0''}{h_0'} = \frac{\mu''}{\mu'} - \frac{\mu'}{\mu} (\gamma + 1).$$

Integrating this equation, one gets

$$h_0' = q_1 \mu^{-(\gamma+1)} \mu', \quad (17)$$

where q_1 is constant. The general solution of this equation depends on γ .

If $\gamma = 0$, then $h_0 = q_1 \ln \mu$, and

$$W(\rho, \dot{\rho}, S) = \varphi(\dot{\rho} \rho^{\beta}, \rho \tilde{S}) + q_1 \ln |\tilde{S}|, \quad (18)$$

and the extension of the kernel is given by the generator

$$-2\beta X_1 + (2\beta + 1)X_3 + 2X_8 - 2\tilde{S}\partial_{\tilde{S}}.$$

Here $\tilde{S} = \mu(S)$. If $\gamma \neq 0$, then $h_0 = q_1 \mu^{-\gamma}$. Changing the function φ (if it is necessary), one can assume that

$$W(\rho, \dot{\rho}, S) = \rho^{\gamma} \varphi(\dot{\rho} \rho^{\beta}, \rho \mu(S)), \quad (19)$$

and the extension of the kernel is given by the generator

$$2(\beta + \gamma)X_1 - (2\beta + \gamma + 1)X_3 - 2X_8 + 2\tilde{S}\partial_S.$$

Now let us suppose that $g_{\rho\rho} = 0$, In this case

$$W(\rho, \dot{\rho}, S) = \varphi(\rho)\psi(\dot{\rho}, S) + h(\rho, S),$$

where

$$\left(\frac{\psi_{\rho\rho\rho}}{\psi_{\rho\rho}}\right)_S \neq 0.$$

Equation (6) becomes

$$k_8 \left(\rho \frac{\varphi'}{\varphi}\right)' = 0. \quad (20)$$

Assume that $\left(\rho \frac{\varphi'}{\varphi}\right)' = 0$ or $\varphi = \rho^\gamma$, where γ is constant. Equation (7) gives

$$\zeta = -\tilde{k}_1 \frac{\dot{\rho}(\psi_{\rho\rho\rho\rho}\psi_{\rho\rho} - \psi_{\rho\rho\rho}^2) + \psi_{\rho\rho}\psi_{\rho\rho\rho}}{\psi_{\rho\rho\rho S}\psi_{\rho\rho} - \psi_{\rho\rho\rho}\psi_{\rho\rho S}}. \quad (21)$$

In this case

$$\tilde{k}_3 = k_8\gamma + \tilde{k}_1\Phi,$$

where

$$\Phi = \frac{\dot{\rho}(\psi_{\rho\rho\rho\rho}\psi_{\rho\rho S} - \psi_{\rho\rho\rho S}\psi_{\rho\rho\rho}) + \psi_{\rho\rho\rho}\psi_{\rho\rho S}}{\psi_{\rho\rho\rho S}\psi_{\rho\rho} - \psi_{\rho\rho\rho}\psi_{\rho\rho S}}. \quad (22)$$

If Φ is not constant, then $\tilde{k}_1 = 0$, and Equation (3) becomes

$$k_8(h_{\rho\rho\rho}\rho - (\gamma - 2)h_{\rho\rho}) = 0.$$

Since, for the existence of an extension of the kernel, one has to consider $k_8 \neq 0$, then

$$h_{\rho\rho} = \mu\rho^{(\gamma-2)}, \quad (23)$$

where $\mu = \mu(S)$.

If $\gamma = 1$, then $h(\rho, S) = \mu(S)\rho \ln|\rho| + \mu_1(S)$. Equation (4) leads to the condition that

μ_1 is constant: in this case it can be assumed without loss of the generality that $\mu_1 = 0$. Thus,

$$W(\rho, \dot{\rho}, S) = \rho(\psi(\dot{\rho}, S) + \mu(S) \ln|\rho|),$$

and the extension of the kernel is given by the generator

$$X_1 - X_3 - X_8.$$

If $\gamma = 0$, then $h(\rho, S) = \mu(S) \ln|\rho| + \mu_1(S)$. Equation (4) leads to $\mu = k$, where k is constant. Thus,

$$W(\rho, \dot{\rho}, S) = \psi(\dot{\rho}, S) + k \ln|\rho| + \mu_1(S),$$

and the extension of the kernel is given by the generator

$$X_3 + 2X_8.$$

If $(\gamma - 1)\gamma \neq 0$, then $h(\rho, S) = \mu(S)\rho^\gamma + \mu_1(S)$. Here one can assume that $\mu = 0$. Equation (4) leads to the finding that μ_1 is constant. Thus,

$$W(\rho, \dot{\rho}, S) = \rho^\gamma \psi(\dot{\rho}, S),$$

and the extension of the kernel is given by the generator

$$2\gamma X_1 - (\gamma + 1)X_3 - 2X_8.$$

Assume that $\Phi = k$ is constant. Then

$$\zeta = -\tilde{k}_1 \frac{\dot{\rho}\psi_{\rho\rho\rho} + k\psi_{\rho\rho}}{\psi_{\rho\rho S}}, \quad \tilde{k}_3 = k_8\gamma + \tilde{k}_1k.$$

Notice that in this case

$$\frac{\dot{\rho}\psi_{\rho\rho\rho} + k\psi_{\rho\rho}}{\psi_{\rho\rho S}}$$

depends on S , say $\lambda(S)$:

$$\frac{\dot{\rho}\psi_{\rho\rho\rho} + k\psi_{\rho\rho}}{\psi_{\rho\rho S}} = \lambda(S).$$

Since for $\lambda(S) = 0$ one obtains

$\Psi_{\rho\rho} = \alpha(S)\dot{\rho}^{-k}$, which leads to a zero denominator in (22), one has to assume that $\lambda(S) \neq 0$. In this case $\Psi(\dot{\rho}, S) = \dot{\rho}^{2-k}\varphi(\dot{\rho}\mu(S))$, where $\mu(S)$ is related with $\lambda(S)$ and $\mu' \neq 0$.

Substituting $\psi(\dot{\rho}, S)$ into (3), one finds

$$k_8\mu'(\rho h_{\rho\rho\rho} - \gamma h_{\rho\rho} + 2h_{\rho\rho}) + \tilde{k}_1(\mu h_{\rho\alpha S} - k\mu' h_{\rho\rho} + 2\mu' h_{\rho\rho}) = 0. \quad (24)$$

Assuming that $\rho h_{\rho\rho\rho} - \gamma h_{\rho\rho} + 2h_{\rho\rho} = 0$, one finds one finds $h_{\rho\rho} = \mu_2(S)\rho^{\gamma-2}$.

For $\gamma=1$, one has

$$h(\rho, S) = \mu_2(S)\rho \ln |\rho| + \mu_3(S).$$

Equation (24) gives

$$\mu\mu'_2 + (2-k)\mu'\mu_2 = 0.$$

The general solution of this equation is

$$\mu_2 = q_1\mu^{k-2}.$$

Substituting it into Equation (24), one gets

$$k_8\mu'^2\mu'_3 + \tilde{k}_1(\mu\mu''\mu'_3 + \mu'^2\mu'_3(k-3) - \mu'\mu\mu''_3) = 0.$$

If μ_3 is constant, then without loss of generality one can assume that $\mu_3 = 0$. Thus,

$$W(\rho, \dot{\rho}, S) = \rho(\dot{\rho}^{2-k}\varphi(\dot{\rho}\mu(S)) + q_1\mu^{k-2}(S)\ln |\rho|),$$

and the extension of the kernel is given by the generators

$$-X_1 + 2X_3 + X_8, 2(1-k)X_1 + kX_3 + 2\tilde{S}\partial_{\tilde{S}}.$$

If $\mu'_3 \neq 0$, then $k_8 = v\tilde{k}_1$, where

$$v = \frac{(\mu\mu''\mu'_3 + \mu'^2\mu'_3(k-3) - \mu'\mu\mu''_3)}{\mu'^2\mu'_3}$$

is constant. Hence, if $\mu_3 = q_2 \ln \mu$, then $v = k-2$, and if $\mu_3 = q_2\mu^\alpha$, where $\alpha \neq 0$, and then $v = k-2-\alpha$. Thus, for

$$W(\rho, \dot{\rho}, S) = \rho(\dot{\rho}^{2-k}\varphi(\dot{\rho}\tilde{S}) + q_1\tilde{S}^{k-2}\ln |\rho|) + q_2\ln \tilde{S},$$

the extension of the kernel is given by the generator

$$-2X_1 + (4-k)X_3 + 2(2-k)X_8 + 2\tilde{S}\partial_{\tilde{S}},$$

and for

$$W(\rho, \dot{\rho}, S) = \rho(\dot{\rho}^{2-k}\varphi(\dot{\rho}\tilde{S}) + q_1\tilde{S}^{k-2}(S)\ln |\rho|) + q_2\tilde{S}^\alpha,$$

the admitted generator is

$$2(\alpha-k+1)X_1 + (k-2\alpha)X_3 - 2\alpha X_8 + 2\tilde{S}\partial_{\tilde{S}}.$$

Assuming that $\gamma = 0$, one has

$$h(\rho, S) = \mu_2(S)\ln |\rho| + \mu_3(S).$$

Equation (24) gives

$$\mu\mu'_2 + (2-k)\mu'\mu_2 = 0.$$

The general solution of this equation is

$$\mu_2 = q_1\mu^{k-2}.$$

Substituting it into Equation (24), one gets

$$k_8\mu^k\mu'^3(2-k) + \tilde{k}_1\mu^3(\mu'\mu'_3\mu + \mu'^2\mu'_3(k-3) - \mu'\mu\mu''_3) = 0.$$

The study of this equation is similar to the previous case. If μ_3 is constant, then without loss of generality one can assume that $\mu_3 = 0$. Thus,

$$W(\rho, \dot{\rho}, S) = \dot{\rho}^{2-k}\varphi(\dot{\rho}\tilde{S}) + \tilde{S}^{k-2}\dot{\rho}^{-k}\ln |\rho|,$$

and the extension of the kernel is given by the generators

$$(1-k)X_1 + kX_3 + \tilde{S}\partial_{\tilde{S}}, X_3 + 2X_8.$$

If $\mu'_3 \neq 0$, then $k_8 = v\tilde{k}_1$, where

$$v = \frac{\mu''\mu'_3\mu + \mu'^2\mu'_3(k-3) - \mu'\mu\mu''_3}{\mu^k\mu'^3(k-2)}$$

is constant. Hence, if $\mu_3 = q_2 \ln \mu$, then $v = k - 2$, and if $\mu_3 = q_2 \mu^\alpha$, where $\alpha \neq 0$, then $v = k - 2 - \alpha$. Thus,

$$W(\rho, \dot{\rho}, S) = \dot{\rho}^{2-k} \varphi(\dot{\rho} \tilde{S}) + q_1 \ln |\rho| (\tilde{S}^{k-2} + 1),$$

and the extension of the kernel is given by the generator

$$2(1-k)X_1 + (k + \mu^{2-k})X_3 + 2\mu^{2-k}X_8 + 2S\partial_{\tilde{S}},$$

and for

$$W(\rho, \dot{\rho}, S) = \dot{\rho}^{2-k} \varphi(\dot{\rho} \mu(S)) + q_1 \tilde{S}^{k-2} (\ln |\rho| + \tilde{S}^{-\alpha}),$$

the admitted generator is

$$2\mu^\alpha(k-2)(1-k)X_1 + (\mu^\alpha k(k-2) - \alpha(\alpha-k+2))X_3 + 2\alpha(k-2-\alpha)X_8 + 2\mu^\alpha(k-2)\tilde{S}\partial_{\tilde{S}}.$$

Assume that $\gamma(\gamma-1) \neq 0$, one has

$$h = \mu_2(S)\rho^\gamma + \mu_3(S),$$

Equation (24) gives

$$\mu\mu'_2 + (2-k)\mu'_2\mu_2 = 0.$$

The general solution of this Equation is

$$\mu_2 = q_1 \mu^{k-2}.$$

Substituting it into Equation (24), one gets

$$k_8 \mu'^2 \mu_3 \gamma + \tilde{k}_1 (\mu'' \mu'_3 \mu + \mu'^2 \mu'_3 (k-3) - \mu' \mu''_3 \mu) = 0.$$

If μ_3 is constant, then without loss of generality one can assume that $\mu_3 = 0$. Thus,

$$W(\rho, \dot{\rho}, S) = \dot{\rho}^{2-k} \varphi(\dot{\rho} S) + q_1 \rho^\gamma \tilde{S}^{k-2},$$

and the extension of the kernel is given by the generators

$$2(1-k)X_1 + kX_3 + 2\tilde{S}\partial_{\tilde{S}}, \quad 2\lambda X_1 - (\gamma-1)X_3 -$$

If $\mu'_3 \neq 0$ then where $k_8 = -v \tilde{k}_1$, where

$$v = \frac{\mu'' \mu'_3 \mu + \mu'^2 \mu'_3 (k-3) - \mu' \mu''_3 \mu}{\mu'^2 \mu'_3 \gamma}$$

is constant. Hence, if $\mu_3 = q_1 \ln \mu$, then $v = (k-2)/\gamma$, and if $\mu_3 = q_1 \mu^\alpha$, where $\alpha \neq 0$, then $v = (k-2-\alpha)/\gamma$. Thus, for

$$W(\rho, \dot{\rho}, S) = \rho^\gamma (\dot{\rho}^{2-k} \varphi(\dot{\rho} \tilde{S}) + q_1 (\tilde{S}^{k-2} + \ln |\tilde{S}|)),$$

the extension of the kernel is given by the generator

$$2\gamma X_1 - (2\gamma - k + 2)X_3 - 2(2-k)X_8 - 2\tilde{S}\partial_{\tilde{S}},$$

and for

$$W(\rho, \dot{\rho}, S) = \rho^\gamma \dot{\rho}^{2-k} \varphi(\dot{\rho} \tilde{S}) + q_1 \tilde{S}^{k-2} (\rho^\gamma + \tilde{S}^{-\sigma}),$$

the admitted generator is

$$2\gamma(\alpha-k+1)X_1 + (\gamma k - \alpha(\gamma+1))X_3 - 2\alpha X_8 + 2\gamma$$

Assuming that $\rho h_{\rho\rho\rho} - \gamma h_{\rho\rho} + 2h_{\rho\rho} \neq 0$, Equation (24) becomes

$$k_8 = -\tilde{k}_1 \frac{\mu h_{\rho\rho S} - \mu' h_{\rho\rho}(k-2)}{\mu'(\rho h_{\rho\rho\rho} + \gamma h_{\rho\rho}(2-\gamma))}.$$

Since $\tilde{k}_1 \neq 0$ for the existence of the extension of the kernel, then

$$\beta = \frac{\mu h_{\rho\rho S} - \mu' h_{\rho\rho}(k-2)}{\mu'(\rho h_{\rho\rho\rho} + h_{\rho\rho}(2-\gamma))}$$

is constant. Hence, the function $\varphi = h_{\rho\rho}$ satisfies the equation

$$\mu \varphi_S - \beta \mu' \rho \varphi_\rho = \mu' [\beta(2-\gamma) + k-2] \varphi.$$

The general solution of this equation is

$$h(\rho, S) = \mu^{(k-2-\beta\gamma)} f(\rho \mu^\beta) + \mu_3(S).$$

Substituting h into (4), one has

$$\mu \dot{\mu}' \ddot{\mu}_3 = \mu \dot{\mu}_3 \ddot{\mu} + (k - \beta\gamma - 3) \dot{\mu}'^2 \ddot{\mu}_3. \quad (25)$$

The general solution of this equation is

$$\mu_3 = \begin{cases} q_1 \ln \mu & k = \beta\gamma + 2, \\ q_1 \mu^{k - \beta\gamma - 2} & k \neq \beta\gamma + 2. \end{cases}$$

If $\mu_3 = q_1 \ln \mu$, then $k = \beta\gamma + 2$ Thus,

$$W(\rho, \dot{\rho}, S) = \rho^\gamma \dot{\rho}^{-\beta\gamma} \varphi(\rho, \tilde{S}) + q_1 \ln |\tilde{S}| + f(\rho \tilde{S}^\beta),$$

and the extension of the kernel is given by the generator

$$2X_1 - (2 - \beta)X_3 + 2\beta X_8 - 2\tilde{S}\partial_{\tilde{S}},$$

If $\mu_3 = q_1 \mu^{k - \beta\gamma - 2}$, then $k \neq \beta\gamma + 2$ Thus,

$$W(\rho, \dot{\rho}, S) = \rho^\gamma \dot{\rho}^{(k - \beta\gamma - 2)} \varphi(\rho, \tilde{S}) + \tilde{S}^{k - \beta\gamma - 2} (q_1 + f(\rho \tilde{S}^\beta)),$$

the admitted generator is

$$2(\beta\gamma - k + 1)X_1 + (k - \beta(\gamma + 1))X_3 - 2\beta X_8 + 2\tilde{S}\partial_{\tilde{S}}.$$

Now continue the study starting from Equation (20), where $(\frac{\rho\dot{\rho}'}{\dot{\rho}})' \neq 0$. In this case $k_8 = 0$,

$$\tilde{k}_3 = \tilde{k}_1 \Phi,$$

where Φ and ξ are defined by (22) and (21)

$$\Phi = \frac{\dot{\rho}(\Psi_{\rho\rho\rho\rho}\Psi_{\rho\rho S} - \Psi_{\rho\rho\rho S}\Psi_{\rho\rho\rho}) + \Psi_{\rho\rho\rho}\Psi_{\rho\rho S}}{\Psi_{\rho\rho S}\Psi_{\rho\rho} - \Psi_{\rho\rho\rho}\Psi_{\rho\rho S}},$$

$$\xi = -\tilde{k}_1 \frac{\dot{\rho}(\Psi_{\rho\rho\rho\rho}\Psi_{\rho\rho} - \Psi_{\rho\rho\rho}^2) + \Psi_{\rho\rho}\Psi_{\rho\rho\rho}}{\Psi_{\rho\rho S}\Psi_{\rho\rho} - \Psi_{\rho\rho\rho}\Psi_{\rho\rho S}}.$$

If Φ is not constant, then $\tilde{k}_1 = 0$ and in this case there is no extension of the kernel. This means that Φ is constant, say k . Then

$$\xi = -\tilde{k}_1 \frac{\dot{\rho}\Psi_{\rho\rho\rho} + k\Psi_{\rho\rho}}{\Psi_{\rho\rho S}}, \quad \tilde{k}_3 = \tilde{k}_1 k.$$

Since $\tilde{k}_1 \neq 0$, one has

$$\frac{\dot{\rho}\Psi_{\rho\rho\rho} + k\Psi_{\rho\rho}}{\Psi_{\rho\rho S}} = \lambda(S).$$

The characteristic system of this Equation is

$$\frac{d\dot{\rho}}{\dot{\rho}} = \frac{dS}{-\lambda(S)} = \frac{d\Psi_{\rho\rho}}{-k\Psi_{\rho\rho}} \quad (26)$$

Since for $\lambda(S) = 0$ one obtains $\Psi_{\rho\rho} = \alpha(S)$ $\beta(\dot{\rho})$, which leads to a zero denominator in (22), one has to assume that $\lambda(S) \neq 0$. In this case $\psi(\dot{\rho}, S) = \dot{\rho}^{2-k} \varphi(\dot{\rho}\mu(S))$, where $\mu(S)$ is related with $\lambda(S)$ and $\dot{\mu} \neq 0$.

Equation (3) becomes

$$\mu h_{\rho\rho S} - \dot{\mu}'(k - 2)h_{\rho\rho} = 0.$$

and $h = v\mu^{k-2}$, where v is constant. Thus,

$$W(\rho, \dot{\rho}, S) = \dot{\rho}^{2-k} q \varphi(\rho, \tilde{S}) + v \tilde{S}^{k-2},$$

and the admitted generator is

$$2(1 - k)X_1 + kX_3 + 2\tilde{S}\partial_{\tilde{S}}.$$

Result of the Group Classification

The result of group classification of the Equations (1) is summarized in Table 1. The linear part with respect to $\dot{\rho}$ of the function $W(\rho, \dot{\rho}, S)$ is omitted.

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References

- Gavrilyuk, S.L. and Teshukov, V.M. (2001). Generalized vorticity for bubbly liquid and dispersive shallow water equations. Continuum Mech. Therm., 13:365-382.
- Golubitsky, M. and Stewart, I. (2002). The Symmetry Perspective: from Equilibrium to Chaos in Phase Space and Physical Space. Birkhäuser, Basel, Switzerland, 325p.

Table 1. Result of the group classification

	$W(\rho, \dot{\rho}, S)$	Extensions	Remarks
M_1	$\psi(\rho, \dot{\rho})S$	$2(X_1 - X_3) - 2S\partial_S$	
M_2	$\varphi(\dot{\rho}\rho^\beta, \rho S) + q_1 \ln S $	$2\beta X_1 - (2\beta + 1)X_3 - 2X_8 + 2S\partial_S$	
M_3	$\rho^\gamma \varphi(\dot{\rho}\rho^\beta, \rho S)$	$2(\beta + \gamma)X_1 - (2\beta + \gamma + 1)X_3$ $- 2X_8 + 2S\partial_S$	$\gamma \neq 0$
M_4	$\rho(\psi(\dot{\rho}, S) + S \ln \rho)$	$X_1 - X_3 - X_8$	
M_5	$\psi(\dot{\rho}, S) + k \ln \rho + S$	$X_3 + 2X_8$	
M_6	$\rho^\gamma \psi(\dot{\rho}, S)$	$2\gamma X_1 - (\gamma + 1)X_3 - 2X_8$	$\gamma(\gamma - 1) \neq 0$
M_7	$\rho(\dot{\rho}^{2-k} \varphi(\dot{\rho}S) + q_1 \mu^{k-2} S \ln \rho)$	$X_1 - 2X_3 - X_8,$ $2(1-k)X_1 + kX_3 + 2S\partial_S$	
M_8	$\rho(\dot{\rho}^{2-k} \varphi(\dot{\rho}S) + q_1 \mu^{k-2} S \ln \rho)$ $+ q_2 \ln S $	$2X_1 - (4-k)X_3 - 2(2-k)X_8 - 2S\partial_S$	
M_9	$\rho(\dot{\rho}^{2-k} \varphi(\dot{\rho}S) + q_1 \mu^{k-2} S \ln \rho)$ $+ q_2 S^\alpha$	$2(\alpha - k + 1)X_1 + (k - 2\alpha)X_3$ $- 2\alpha X_8 + 2S\partial_S$	
M_{10}	$\dot{\rho}^{2-k} \varphi(\dot{\rho}S) + S^{k-2} \dot{\rho}^{-k} \ln \rho $	$(1-k)X_1 + kX_3 + S\partial_S,$ $X_3 + 2X_8$	
M_{11}	$\dot{\rho}^{2-k} \varphi(\dot{\rho}S) + q_1 \ln \rho(S^{k-2} + 1)$	$2(1-k)X_1 + kX_3 + 2S\partial_S,$ $2\lambda X_1 - (\gamma - 1)X_3 - 2X_8$	
M_{12}	$\dot{\rho}^{2-k} \varphi(\dot{\rho}S) + q_1 S^{k-2} (\ln \rho + S^{-\alpha})$	$2\mu^\alpha (k - 2)(1 - k)X_1 + (\mu^\alpha k(k - 2)$ $- \alpha(\alpha - k + 2))X_3 + 2\alpha(k - 2 - \alpha)X_8$ $+ 2\mu^\alpha (k - 2)S\partial_S$	
M_{13}	$\dot{\rho}^{2-k} \varphi(\dot{\rho}S) + q_1 \rho^\gamma S^{k-2}$	$2(1-k)X_1 + kX_3 + 2\tilde{S}\partial_S,$ $2\lambda X_1 - (\gamma - 1)X_3 - 2X_8$	
M_{14}	$\rho^\gamma (\dot{\rho}^{2-k} \varphi(\dot{\rho}S) + q_1 (S^{k-2} + \ln S))$	$2\gamma X_1 - (2\gamma - k + 2)X_3$ $- 2(2 - k)X_8 - 2\gamma S\partial_S$	
M_{15}	$\rho^\gamma \dot{\rho}^{2-k} \varphi(\dot{\rho}S) + q_1 S^{k-2} (\rho^\gamma + S^{-\alpha})$	$2\gamma(\alpha - k + 1)X_1 + (\gamma k - \alpha(\gamma + 1))X_3$ $- 2\alpha X_8 + 2\gamma S\partial_S$	
M_{16}	$\rho^\gamma \dot{\rho}^{-\beta\gamma} \varphi(\rho, S) + q_1 \ln S + f(\rho S^\beta)$	$2X_1 - (2 - \beta)X_3 + 2\beta X_8 - 2S\partial_S$	
M_{17}	$\rho^\gamma \dot{\rho}^{(k-\beta\gamma-2)} \varphi(\rho, S)$ $+ S^{(k-\beta\gamma-2)} (q_1 + f(\rho S^\beta))$	$2(\beta\gamma - k + 1)X_1 + (k - \beta(\gamma + 1))X_3$ $- 2\beta X_8 + 2S\partial_S$	
M_{18}	$\dot{\rho}^{2-k} q \varphi(\rho, S) + \nu S^{k-2}$	$2(1-k)X_1 + kX_3 + 2S\partial_S$	

Green, A.E. and Naghdi, P.M. (1976). A derivation of equations for wave propagation in water of variable depth. J. Fluid Mech., 78:237-246.

Hematulin, A., Meleshko, S.V., and Gavriluk, S.G. (2007). Group classification of one-dimensional equations of fluids with internal inertia. Math. Method. Appl. Sci., 30:2101-2120.

Ibragimov, N.H. (1994). CRC Handbook of Lie Group Analysis of Differential Equations. Volume 1: Symmetries, Exact Solutions and Conservation

Laws. 1st ed. CRC Press, Boca Raton, FL, USA, 430p.

Ibragimov, N.H. (1995). CRC Handbook of Lie Group Analysis and Ordinary Differential Equations. Volume 2: Applications in Engineering and Physical Sciences. 1st ed. CRC Press, Boca Raton, FL, USA, 546p.

Ibragimov, N.H. (1996). CRC Handbook of Lie Group Analysis and Ordinary Differential Equation. Volume 3: New Trends in Theoretical

- Developments and Computational Methods. 1st ed. CRC Press, Boca Raton, FL, USA, 536p.
- Ibragimov, N.H. (1999). Elementary Lie Group Analysis and Ordinary Differential Equations. Wiley & Sons, Chichester, UK, 341p.
- Iordanski, S.V. (1960). On the equations of motion of liquid containing gas bubbles. Zh. Prikl. Mekh. Tekhn. Fiz., 3: 102-111p.
- Kogarko, B.S. (1961). On the model of cavitating liquid. Dokl Akad. Nauk. SSSR, 137:1331-1333.
- Marsden, J. and Ratiu T. (1994). Introduction to Mechanics and Symmetry. 2nd ed. Springer-Verlag, NY, USA, 555p.
- Olver, P.J. (1993). Applications of Lie Groups to Differential Equations. 2nd ed. Springer-Verlag, NY, USA, 516p.
- Ovsiannikov, L.V. (1994). Program SUBMODELS. Gas dynamics. J. Appl. Math. Mech-USS., 58(4):30-55.
- Ovsiannikov, L.V. (1978). Group analysis of differential equations. 1st ed. Academic Press, NY, USA, (translated from Russian), 416p.
- Ovsiannikov, L.V. (1999). Some results of the implementation of the "PODMODELI" program for the gas dynamics equations. J. Appl. Math. Mech-USS., 63(3):349-358.
- Salmon, R. (1998). Lectures on Geophysical Fluid Dynamics. 1st ed. Oxford University Press, NY, USA, 1013p.
- Siriwat, P. and Meleshko, S.V. (2008). Applications of group analysis to the three-dimensional equations of fluids with internal inertia. SIGMA 4 (2008), 027:1-19.
- Siriwat, P. and Meleshko, S.V. (2011). Group classification of one-dimensional non-isentropic equations of fluids with internal inertia. Continuum Mech. Therm., (In press).
- Wijngaarden, van, L. (1968). On the equations of motion for mixtures of liquid and gas bubbles. J. Fluid Mech., 33:465-474.