

PARALLEL ALGORITHMS FOR NUMERICAL SIMULATION OF TURBULENT WAKE DYNAMICS IN LINEAR STRATIFIED FLUID

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Abstract

The aim of this research is to apply the parallel computation techniques of functional and domain decompositions to a numerical simulation of turbulent wake dynamics behind an axisymmetric body in a stratified fluid. A theoretical estimate of speedup is constructed. The dependence of speedup on the latency and bandwidth of parallel computer systems is analyzed. The parallel numerical results have been compared with the experimental and numerical data for momentumless and drag turbulent wakes in a homogeneous and stratified fluid.

Keywords: Parallel computing, functional and domain decomposition, splitting method, turbulent wake, stratified fluid

Introduction

Research on stratified turbulent wakes has both meteorological (flow over islands, mountains, or sea-mountains) and naval applications (submerged bodies). The turbulent wake behind towed and self-propelled bodies in a homogeneous and stratified fluid has been considered in a number of theoretical, experimental, and computational works (Lin and Pao, 1979; Hassid, 1980; Chernykh and Voropayeva, 1999; Voropayeva *et al.*, 2000; Chernykh *et al.*, 2006; Meunier and Spedding, 2006). Based on a three-dimensional parabolized system of averaged equation for motion, continuity, incompressibility, turbulent energy, and the rate of dissipation, a

numerical modeling of turbulent wake dynamics has been conducted by Chernykh and Voropayeva (1999), Voropayeva *et al.* (2000), and Chernykh *et al.* (2006). To predict the average characteristics of a turbulent wake can take a long time on a serial computer. So the objectives of recent research are to develop and analyze parallel algorithms that improve efficiency and decrease the computational time of numerical models considered in Chernykh and Voropayeva (1999) and Chernykh *et al.* (2006). Both functional and domain decomposition techniques are used to develop parallel algorithms in the present study.

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The Governing Equations

The equations which govern the flow in a far turbulent wake of a body of revolution in a stable stratified medium are the following parabolized system in the Oberbeck-Boussineq approach

$$U_0 \frac{\partial U_d}{\partial x} + V \frac{\partial U_d}{\partial y} + W \frac{\partial U_d}{\partial z} = \frac{\partial}{\partial y} \langle u'v' \rangle + \quad (1)$$

$$\frac{\partial}{\partial z} \langle u'w' \rangle,$$

$$U_0 \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + W \frac{\partial V}{\partial z} = -\frac{1}{\rho_0} \frac{\partial \langle p_1 \rangle}{\partial y} - \quad (2)$$

$$\frac{\partial}{\partial y} \langle v'^2 \rangle - \frac{\partial}{\partial z} \langle v'w' \rangle,$$

$$U_0 \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} = -\frac{1}{\rho_0} \frac{\partial \langle p_1 \rangle}{\partial z} - \quad (3)$$

$$\frac{\partial}{\partial y} \langle v'w' \rangle - \frac{\partial}{\partial z} \langle w'^2 \rangle - g \frac{\langle \rho_1 \rangle}{\rho_0},$$

$$U_0 \frac{\partial \langle \rho_1 \rangle}{\partial x} + V \frac{\partial \langle \rho_1 \rangle}{\partial y} + W \frac{\partial \langle \rho_1 \rangle}{\partial z} + W \frac{d\rho_s}{dz} = \quad (4)$$

$$-\frac{\partial}{\partial y} \langle v'\rho' \rangle - \frac{\partial}{\partial z} \langle w'\rho' \rangle,$$

$$\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \frac{\partial U_d}{\partial x}, \quad (5)$$

where U_0 is the free stream velocity; $U_d = U_0 - U$ is the defect of the mean stream wise velocity component; U, V, W are velocity components of the mean flow in the direction of the axes x, y, z ; p_1 is the deviation of the pressure from the hydrostatic one conditioned by the stratification ρ_s ; g is the gravity acceleration; $\langle \rho_1 \rangle$ is the mean density defect: $\rho_1 = \rho - \rho_s$; $\rho_s = \rho_s(z)$ is the undisturbed fluid density; $d\rho_s/dz \leq 0$ (stable

stratification), $\rho_0 = \rho_s(0)$; the dash denotes the pulsation components; and the symbol $\langle \cdot \rangle$ denotes the averaging. The coordinate system is related to the moving body in such a way that the velocity of its motion is equal to $-U_0$, and the z -axis is directed vertically upwards, in the counter-gravity direction. The fluid density is assumed to be a linear function of temperature; the stratification is assumed to be weak. Both small items involving the derivative with respect to the variable x , and the factors in the form of a coefficient of laminar viscosity or diffusion, have been omitted in the right hand sides.

The system of equations (1) - (5) is not closed. We consider below two semi-empirical models, each of them together with equations (1) - (5) forms a closed flow model. In Model 1 the quantities of the components of the Reynolds stress tensor $\langle u'_i u'_j \rangle$ (besides $\langle u'_2 u'_3 \rangle = \langle v'w' \rangle$), of turbulent fluxes $\langle u'_i \rho' \rangle$, and the dispersion of the density fluctuations $\langle \rho'^2 \rangle$ are approximated by the algebraic relations.

$$\frac{\langle u'_i u'_j \rangle}{e} = \frac{2}{3} \delta_{ij} + \frac{1 - c_2}{\varepsilon c_1} \quad (6)$$

$$\left[P_{ij} + G_{ij} - \frac{2}{3} \delta_{ij} (P + G) \right],$$

$$-\langle u'_i \rho' \rangle = \frac{e}{c_{1T} \varepsilon} \left[\langle u'_i u'_k \rangle \frac{\partial \langle \rho \rangle}{\partial x_k} + (1 - c_{2T}) \right] \quad (7)$$

$$\left(\langle u'_k \rho \rangle \frac{\partial U}{\partial x_k} - \frac{g_i}{\rho_0} \langle \rho'^2 \rangle \right) \left] \right],$$

$$\langle \rho'^2 \rangle = -\frac{2}{c_T} \frac{e}{\varepsilon} \langle w' \rho' \rangle \frac{\partial \langle \rho \rangle}{\partial z}, \quad (8)$$

$$P_{ij} = - \left(\langle u'_i u'_k \rangle \frac{\partial U_j}{\partial x_k} + \langle u'_j u'_k \rangle \frac{\partial U_i}{\partial x_k} \right), \quad (9)$$

$$G_{ij} = \frac{1}{\rho_0} \left(\langle u'_i \rho' \rangle g_j + \langle u'_j \rho' \rangle g_i \right),$$

$i, j, k = 1, 2, 3$; and

$$\bar{g} = (g_1, g_2, g_3) = (0, 0, -g),$$

$$2G = G_{ii}, \quad U_1 = U, \quad U_2 = V, \quad U_3 = W$$

Here and below the summation convention is assumed. By analogy with Hassid (1980), let us simplify the expression (6) - (9) with regard to the physical peculiarities of the flow under consideration – a jet turbulent co-flow in the field of the gravity force at a large distance from the body. Expressions (6) - (8) are simplified as follows:

$$\langle u'v' \rangle = \frac{1-c_2}{c_1} \frac{e \langle v'^2 \rangle}{\varepsilon} \frac{\partial U_d}{\partial y} = K_y \frac{\partial U_d}{\partial y}, \quad (10)$$

$$\langle u'w' \rangle = K_z \frac{\partial U_d}{\partial z}, \quad (11)$$

$$\langle \rho'^2 \rangle = -\frac{2}{c_T} \frac{e}{\varepsilon} \langle w' \rho' \rangle \frac{\partial \langle \rho \rangle}{\partial z},$$

$$-\langle u' \rho' \rangle = \frac{1}{c_{1T}} \frac{e}{\varepsilon} \left[\langle u'w' \rangle \frac{\partial \langle \rho \rangle}{\partial z} + \right. \quad (12)$$

$$\left. (1-c_{2T}) \langle w' \rho' \rangle \frac{\partial U}{\partial z} \right],$$

$$-\langle v' \rho' \rangle = \frac{1}{c_{1T}} \frac{e}{\varepsilon} \langle v'^2 \rangle \frac{\partial \langle \rho \rangle}{\partial y} = K_{\rho y} \frac{\partial \langle \rho \rangle}{\partial y}, \quad (13)$$

$$-\langle w' \rho' \rangle = K_{\rho z} \frac{\partial \langle \rho \rangle}{\partial z}, \quad (14)$$

$$K_{\rho z} = \frac{e \langle w'^2 \rangle}{c_{1T} \varepsilon \left(1 - 2 \frac{1-c_{2T}}{c_{1T} c_T} \frac{g}{\rho_0} \frac{e^2}{\varepsilon^2} \frac{\partial \langle \rho \rangle}{\partial z} \right)}, \quad (15)$$

$$K_z = \frac{\left[(1-c_2) e \langle w'^2 \rangle - \frac{(1-c_2)(1-c_{2T})}{c_{1T}} \frac{e^2}{\varepsilon} \frac{g}{\rho_0} \langle w' \rho' \rangle \right]}{c_1 \varepsilon \left(1 - \frac{(1-c_2)}{c_{1T}} \frac{g}{\rho_0} \frac{e^2}{\varepsilon^2} \frac{\partial \langle \rho \rangle}{\partial z} \right)}.$$

To determine the values of the turbulence energy e , the rate of dissipation ε , and the shear Reynolds stress $\langle v'w' \rangle$, we make use of the differential equations.

$$U_0 \frac{\partial e}{\partial x} + V \frac{\partial e}{\partial y} + W \frac{\partial e}{\partial z} = \frac{\partial}{\partial y} K_{ey} \frac{\partial e}{\partial y} + \quad (16)$$

$$\frac{\partial}{\partial z} K_{ez} \frac{\partial e}{\partial z} + P + G - \varepsilon,$$

$$U_0 \frac{\partial \varepsilon}{\partial x} + V \frac{\partial \varepsilon}{\partial y} + W \frac{\partial \varepsilon}{\partial z} = \frac{\partial}{\partial y} K_{ey} \frac{\partial \varepsilon}{\partial y} + \frac{\partial}{\partial z} \quad (17)$$

$$K_{ez} \frac{\partial \varepsilon}{\partial z} + c_{\varepsilon 1} \frac{\varepsilon}{e} (P + G) - c_{\varepsilon 2} \frac{\varepsilon^2}{e},$$

$$U_0 \frac{\partial \langle v'w' \rangle}{\partial x} + V \frac{\partial \langle v'w' \rangle}{\partial y} + W \frac{\partial \langle v'w' \rangle}{\partial z} = \frac{\partial}{\partial y} \quad (18)$$

$$K_{ey} \frac{\partial \langle v'w' \rangle}{\partial y} + \frac{\partial}{\partial z} K_{ez} \frac{\partial \langle v'w' \rangle}{\partial z} + (1-c_2)$$

$$G_{23} - c_1 \frac{\varepsilon}{e} \langle v'w' \rangle, P_{23} = - \left(\langle v'^2 \rangle \frac{\partial W}{\partial y} + \langle w'^2 \rangle \frac{\partial V}{\partial z} \right),$$

$$G_{23} = -\frac{g}{\rho_0} \langle v' \rho' \rangle.$$

The turbulent viscosity coefficients in equations (16) - (18) are expressed in the form $K_{ey} = K_y$, $K_{ez} = K_z$, $K_{\rho y} = K_{ey} / \sigma$, and $K_{\rho z} = K_{ez} / \sigma$. As a result of the above simplifications, Model 1 of the far turbulent wake represents a system of differential equations (1) - (5), (10) - (18).

In Model 2, the values $\langle u_i'^2 \rangle$ ($i = 1, 2, 3$) are calculated by solving the corresponding transport differential equations

$$U_0 \frac{\partial \langle u'^2 \rangle}{\partial x} + V \frac{\partial \langle u'^2 \rangle}{\partial y} + W \frac{\partial \langle u'^2 \rangle}{\partial z} \quad (19)$$

$$= \frac{\partial}{\partial y} K_{ey} \frac{\partial \langle u'^2 \rangle}{\partial y} + \frac{\partial}{\partial z} K_{ez} \frac{\partial \langle u'^2 \rangle}{\partial z} + P_{11} + G_{11}$$

$$- \frac{2}{3} \varepsilon - C_1 \frac{\varepsilon}{e} \left(\langle u'^2 \rangle - \frac{2}{3} e \right) - C_2 \left(P_{11} - \frac{2}{3} P \right)$$

$$- C_2 \left(G_{11} - \frac{2}{3} G \right),$$

$$U_0 \frac{\partial \langle v'^2 \rangle}{\partial x} + V \frac{\partial \langle v'^2 \rangle}{\partial y} + W \frac{\partial \langle v'^2 \rangle}{\partial z} \quad (20)$$

$$= \frac{\partial}{\partial y} K_{ey} \frac{\partial \langle v'^2 \rangle}{\partial y} + \frac{\partial}{\partial z} K_{ez} \frac{\partial \langle v'^2 \rangle}{\partial z} + P_{22} + G_{22}$$

$$- \frac{2}{3} \varepsilon - C_1 \frac{\varepsilon}{e} \left(\langle v'^2 \rangle - \frac{2}{3} e \right) - C_2 \left(P_{22} - \frac{2}{3} P \right)$$

$$- C_2 \left(G_{22} - \frac{2}{3} G \right),$$

$$U_0 \frac{\partial \langle w'^2 \rangle}{\partial x} + V \frac{\partial \langle w'^2 \rangle}{\partial y} + W \frac{\partial \langle w'^2 \rangle}{\partial z} \quad (21)$$

$$= \frac{\partial}{\partial y} K_{ey} \frac{\partial \langle w'^2 \rangle}{\partial y} + \frac{\partial}{\partial z} K_{ez} \frac{\partial \langle w'^2 \rangle}{\partial z} + P_{33} + G_{33}$$

$$- \frac{2}{3} \varepsilon - C_1 \frac{\varepsilon}{e} \left(\langle w'^2 \rangle - \frac{2}{3} e \right) - C_2 \left(P_{33} - \frac{2}{3} P \right)$$

$$- C_2 \left(G_{33} - \frac{2}{3} G \right),$$

$$K_{ey} = C_s \frac{e}{\varepsilon} \langle v'^2 \rangle, \quad K_{ez} = C_s \frac{e}{\varepsilon} \langle w'^2 \rangle,$$

$$e = (\langle u'^2 \rangle + \langle v'^2 \rangle + \langle w'^2 \rangle) / 2.$$

$$P_{11} = 2 \left(K_y \left(\frac{\partial U_d}{\partial y} \right)^2 + K_z \left(\frac{\partial U_d}{\partial z} \right)^2 \right) = 2P,$$

$$G_{11} = 0, P_{22} = 0, G_{22} = 0, P_{33} = 0,$$

$$G_{33} = -2 \frac{g}{\rho_0} \langle w' \rho' \rangle = 2G, \quad P = \langle u' v' \rangle \frac{\partial U_d}{\partial y}$$

$$+ \langle u' w' \rangle \frac{\partial U_d}{\partial z} = K_y \left(\frac{\partial U_d}{\partial y} \right)^2 + K_z \left(\frac{\partial U_d}{\partial z} \right)^2,$$

$$G = - \frac{g}{\rho_0} \langle w' \rho' \rangle = \frac{g}{\rho_0} K_{\rho z} \frac{\partial \rho}{\partial z}.$$

To study the passive scalar dynamics in turbulent wakes in addition to equations (1) - (5), (10) - (18), and (19) - (21), we use the transport equations for the averaged concentration of the passive scalar Θ and the dispersion of passive scalar concentration fluctuations $\langle \theta'^2 \rangle$.

$$U_0 \frac{\partial \Theta}{\partial x} + V \frac{\partial \Theta}{\partial y} + W \frac{\partial \Theta}{\partial z} = \frac{\partial}{\partial y} K_{ey} \frac{\partial \Theta}{\partial y} + \quad (22)$$

$$\frac{\partial}{\partial z} K_{ez} \frac{\partial \Theta}{\partial z},$$

$$U_0 \frac{\partial \langle \theta'^2 \rangle}{\partial x} + V \frac{\partial \langle \theta'^2 \rangle}{\partial y} + W \frac{\partial \langle \theta'^2 \rangle}{\partial z} = \frac{\partial}{\partial y} \quad (23)$$

$$K_{1ey} \frac{\partial \langle \theta'^2 \rangle}{\partial y} + \frac{\partial}{\partial z} K_{1ez} \frac{\partial \langle \theta'^2 \rangle}{\partial z} -$$

$$2\langle v'\theta' \rangle \frac{\partial \Theta}{\partial y} - 2\langle w'\theta' \rangle \frac{\partial \Theta}{\partial z} - N_\Theta,$$

where

$$K_{\Theta y} = \frac{e\langle v'^2 \rangle}{c_{1T}\mathcal{E}}, K_{\Theta z} = \frac{e\langle w'^2 \rangle}{c_{1T}\mathcal{E}}, K_{1\Theta y} =$$

$$\tilde{c}_\varphi K_{\Theta y}, K_{1\Theta z} = \tilde{c}_\varphi K_{\Theta z}, \langle v'\theta' \rangle = -K_{\Theta y} \frac{\partial \Theta}{\partial y},$$

$$\langle w'\theta' \rangle = -K_{\Theta z} \frac{\partial \Theta}{\partial z}, N_\Theta = c_T \frac{\langle \theta'^2 \rangle \mathcal{E}}{e}.$$

The quantities $c_1, c_2, c_{1T}, c_{2T}, c_T, c_{\varepsilon 1}, c_{\varepsilon 2}, \sigma, \tilde{c}_\varphi$ are empirical constants. Their values are well-known and taken as the following 2.2, 0.55, 3.20, 0.50, 1.25, 1.44, 1.92, 1.30, and 0.42.

The marching variable x in equations (1) - (4), (16) - (18), and (19) - (22) plays the role of time. At the distance $x = x_0$ from the body, the following initial conditions were specified:

$$e(x_0, y, z) = \Phi_1(r), \quad \varepsilon(x_0, y, z) = \Phi_2(r),$$

$$U_d(x_0, y, z) = \Phi_3(r), \quad \Theta(x_0, y, z) = \Phi_4(r),$$

$$\langle \theta'^2 \rangle(x_0, y, z) = \Phi_5(r), \quad r^2 = y^2 + z^2,$$

$$0 \leq r < \infty, \quad \langle \rho_1 \rangle = V = W = \langle v'w' \rangle = 0,$$

$$-\infty < y, z < \infty, \quad x = x_0.$$

Here $\Phi_1(r)$, $\Phi_2(r)$, and $\Phi_3(r)$ are the functions consistent with the experimental data of Lin and Pao (1979) in the homogeneous fluid, and $\Phi_4(r)$ and $\Phi_5(r)$ are compactly supported bell shaped functions. At $r \rightarrow \infty$ the free stream conditions were specified

$$U_d = V = W = \langle \rho_1 \rangle = e = \varepsilon = \Theta = \langle \theta'^2 \rangle = \quad (24)$$

$$\langle v'w' \rangle = 0.$$

From the symmetry considerations, the solution was determined only in the first quadrant of the (y, z) plane using the following boundary conditions:

$$\frac{\partial \langle \rho_1 \rangle}{\partial y} = \frac{\partial e}{\partial y} = \frac{\partial \varepsilon}{\partial y} = V = \frac{\partial W}{\partial y} = \frac{\partial U_d}{\partial y} = \langle v'w' \rangle$$

$$= 0, \quad \frac{\partial \Theta}{\partial y} = \frac{\partial \langle \theta'^2 \rangle}{\partial y} = 0, \quad y = 0, \quad z \geq 0,$$

$$\langle \rho_1 \rangle = \frac{\partial e}{\partial z} = \frac{\partial \varepsilon}{\partial z} = W = \frac{\partial V}{\partial z} = \frac{\partial U_d}{\partial z} = \langle v'w' \rangle =$$

$$0, \quad \frac{\partial \Theta}{\partial z} = \frac{\partial \langle \theta'^2 \rangle}{\partial z} = 0, \quad z = 0, \quad y \geq 0.$$

In the numerical solution of the problem, the boundary conditions corresponding to $r \rightarrow \infty$ were translated to the boundaries of a sufficiently large rectangle $0 \leq y \leq y_*$; $0 \leq z \leq z_*$.

The problem variables can be made dimensionless by using the characteristic length D , the body diameter, the velocity scale U_0 , and the characteristic value of the averaged concentration of the passive scalar $\Theta_0 = \Phi_4(0)$. As a result, the value $4\pi^2 / Fr^2$ will appear in the dimensionless equations instead of g , where Fr is the density Froude number defined as

$$Fr = \frac{U_0 T}{D}, \quad T = \frac{2\pi}{\sqrt{ag}} = \frac{1}{N}, \quad \text{and}$$

$$a = -\frac{1}{\rho_0} \cdot \frac{d\rho_s}{dz} \Big|_{z=0},$$

where T and N are the Brunt-Vaisala period and frequency respectively. For the interpretation of the computational results, it is convenient to introduce the time t related to the distance from the body

$$t = x/U_0, \quad t^* = t/T = xD/U_0DT = x^*/Fr.$$

Computational Details

There are several numerical models of turbulent wake dynamics in a stratified fluid that can successfully be applied (Chernykh and Voropayeva, 1999; Voropayeva *et al.*, 2000; Chernykh *et al.*, 2006) to predict the decay of the turbulent wake behind towed and self-propelled bodies. However, computer implementation of these models on a single processor computer still takes a long time even using those models which have been developed based on the Reynolds-averaged Navier-Stokes approach. These models constitute more than a dozen partial differential equations which couple average characteristics of the turbulent flow. For example, the above Model 2 comprises 12 differential equations. According to the nature of the problems and the characteristics of the numerical models, both the functional and domain decomposition techniques can be used to develop the parallel algorithms for these classes of the problems.

Sequential Algorithm

A sequential Algorithm of the problem solution is based on the implicit splitting into space variables for the equations (1), (4), (16) - (18), and (19) - (22) and based on the application of an explicit splitting into the physical processes to the system of equations (2), (3), and (5). Let all unknown functions be given at the $x = x^n$. The solution procedure consists of the following steps which advanced all unknown functions in the marching direction x :

1. The defect of the mean stream-wise velocity component U_d is computed by evaluating the finite-difference approximation of equation (1).

2. The velocity components of the averaged motion V^{n+1} , W^{n+1} , and the pressure deviation from hydrostatic, $\langle p \rangle^{n+1}$, are computed by using equations (2), (3), and (5). Here we utilize the idea of the splitting method in which the computational process is divided into three stages. The first stage consists in computing the artificial velocity components V^* and W^*

with an explicit approximation of equations (2) and (3). Then in a second stage, we compute $\langle p \rangle^{n+1}$ from the solution to the Poisson equation with the Neumann boundary conditions along the axis of symmetry and the Dirichlet boundary conditions along the far boundaries. The method of the stabilizing correction (Yanenko, 1971) is used in this stage. Finally, in the third stage, the new velocity vector components V^{n+1} and W^{n+1} are computed from the requirement of the vanishing divergence.

3. equations (4), (16) - (18), (19) - (21), and (22) are sequentially integrated with the use of the method of splitting in terms of the spatial variables (Yanenko, 1971).

From the consideration of a simple computer code implementation, we have used the idea of the well-known Gauss-Seidel method. At the computation of the functions $\langle p \rangle^{n+1}$, e^{n+1} , ε^{n+1} and $\langle v'w' \rangle^{n+1}$, we have used the quantities already known at this level $x = x^{n+1}$, and the remaining functions are taken from the previous level $x = x^n$. A flowchart of the sequential algorithm is shown in Figure 1. A more detailed description can be seen in Chernykh *et al.* (2006, 2008).

Parallel Functional Decomposition

The algorithm for solving the problem of turbulent wake dynamics in a stratified fluid is reduced to the successive integration of the transport equations to find the averaged velocity components, turbulent energy, dissipation rate, components of the Reynolds stress tensor, and so on. Suppose we have M jobs J_i for $i = 1, 2, \dots, M$, which are executed in sequential order as shown in Figure 2.

Let t_i be the run time of job J_i for $i = 1, 2, \dots, M$. Hence, the run time of a sequential program T_s is $T_s = \sum_{i=1}^M t_i$. To utilize the parallel

functional decomposition technique, let us assume that the sequential processes can be executed as groups of disjoint tasks as k depicted in Figure 3,

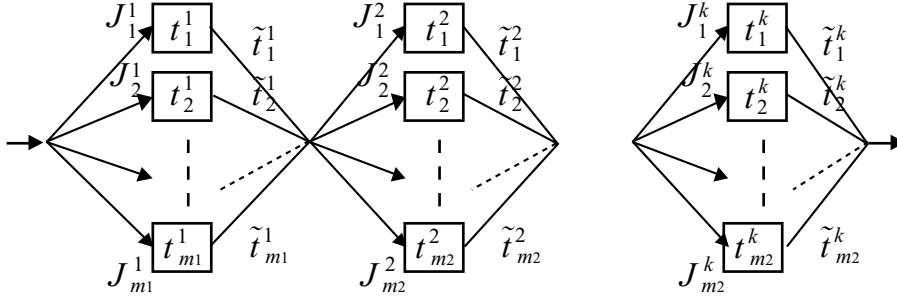


Figure 3. A parallel functional decomposition process

$$T^j \approx \left\lceil \frac{m_j}{P} \right\rceil \max_{1 \leq i \leq m_j} t_i^j \quad \text{and} \quad T_{comm}^j \approx \sum_{i=1}^{m_j} (P-1) \tilde{t}_i^j.$$

Since, $\tilde{t}_i^j = t_i + \frac{MessageSize_i^j}{BandWidth}$ where t_i is the latency time of the parallel network and $MessageSize_i^j$ is the message size of job J_i^j for $j = 1, 2, \dots, k; i = 1, 2, \dots, m_j$. Therefore, an estimate speedup is

$$S = \frac{\sum_{i=1}^M t_i}{\sum_{j=1}^k \left(\left\lceil \frac{m_j}{P} \right\rceil \max_{1 \leq i \leq m_j} t_i^j + \sum_{i=1}^{m_j} (P-1) \left(t_i + \frac{MessageSize_i^j}{Bandwidth} \right) \right)}. \quad (25)$$

For example, let us assume that we have k groups where each group has m jobs, each job uses t seconds for the execution time, and the message size of the data for all jobs is the same, $MessageSize_i^j = MessageSize$. If the term of the communication time is zero, the speedup will be about P times the sequential algorithm, which corresponds to the ideal speedup.

We analyzed the sequential algorithm (Figure 1) and found out that some components can be executed simultaneously in different CPUs. In the case of Model 2, the parallel functional decomposition algorithm can be written in the following form.

1. Initialization to create a group of communication for the Message Passing Interface.
2. The initial and boundary processes.
3. Divide jobs for each processor.
4. Compute for U_d, V^* and W^* simultaneously ($m_1 = 3$).

5. Update data for U_d, V^* and W^* (communication before next stage).
6. Compute for $\langle p_1 \rangle^{n+1}$ by all processors ($m_2 = 1$).
7. Compute for V^{n+1} and W^{n+1} on two processors simultaneously ($m_3 = 2$).
8. Update data for V^{n+1} and W^{n+1} (communication before next stage).
9. Compute for $\langle \rho_1 \rangle^{n+1}, \epsilon^{n+1}, \langle v'w' \rangle^{n+1}, \langle u'^2 \rangle^{n+1}, \langle v'^2 \rangle^{n+1}, \langle w'^2 \rangle^{n+1}$ and Θ^{n+1} simultaneously ($m_4 = 7$).
10. Update data for $\langle \rho_1 \rangle^{n+1}, \epsilon^{n+1}, \langle v'w' \rangle^{n+1}, \langle u'^2 \rangle^{n+1}, \langle v'^2 \rangle^{n+1}, \langle w'^2 \rangle^{n+1}$ and Θ^{n+1} (communication before next stage).
11. Compute for e^{n+1} ($m_5 = 1$).
12. Check conditions for exit loop, or if not, go to step 4.

Parallel Domain Decomposition

In mathematics, numerical analysis, and numerical partial differential equations, domain decomposition methods solve a boundary value problem by splitting it into smaller boundary value problems on sub domains and iterating to coordinate the solution between the sub domains. The problems on the sub domains are independent, which makes domain decomposition methods suitable for parallel computing. In general, "Domain decomposition" refers to any method that divides the original problem domain into pieces and solves it locally on each sub domain.

Models 1 and 2 of turbulent wake dynamics in a stratified fluid involve the solution of several transport equations which are solved by an implicit splitting scheme over

spatial variables, Yanenko (1971). Let us consider as an example the 2D problem in a rectangular domain Ω for a typical transport equation (this can be a differential equation for turbulent energy transport, e , dissipation, ε , and other unknowns).

$$\frac{\partial U}{\partial t} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = \frac{\partial}{\partial y} \left(K_y \frac{\partial U}{\partial y} \right) \quad (26)$$

$$+ \frac{\partial}{\partial z} \left(K_z \frac{\partial U}{\partial z} \right) + F, \quad \text{in } \Omega \times [0, T],$$

$$U(0, y, z) = U^0(0, y, z), \quad (y, z) \in \Omega,$$

$$U(t, y, z) = g(y, z), \quad (y, z) \in \partial\Omega \times [0, T].$$

For the sake of simplicity, assume a uniform tensor-product grid in the domain Ω

$$\Omega_h = \{(y_i, z_j) \mid y_i = i * \Delta y, z_j = j * \Delta z, i = 0, \dots,$$

$$M+1; j = 0, \dots, N+1; \Delta y = 1/M; \Delta z = 1/N\}.$$

The splitting scheme is written in two half time-steps as follows; during the first half step the following discretization is used:

$$\frac{U_{i,j}^{n+1/2} - U_{i,j}^n}{\Delta t} + V_{i,j} \frac{U_{i+1,j}^{n+1/2} - U_{i-1,j}^{n+1/2}}{2\Delta y} =$$

$$\frac{(K_y)_{i+1/2,j} (U_{i+1,j}^{n+1/2} - U_{i,j}^{n+1/2}) - (K_y)_{i-1/2,j} (U_{i,j}^{n+1/2} - U_{i-1,j}^{n+1/2})}{\Delta y^2} + F_{i,j},$$

and during the second half step:

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n+1/2}}{\Delta t} + W_{i,j} \frac{U_{i,j+1}^{n+1/2} - U_{i,j-1}^{n+1/2}}{2\Delta z} =$$

$$\frac{(K_z)_{i,j+1/2} (U_{i,j+1}^{n+1/2} - U_{i,j}^{n+1/2}) - (K_z)_{i,j-1/2} (U_{i,j}^{n+1/2} - U_{i,j-1}^{n+1/2})}{\Delta z^2}.$$

The equation of the first half step can be written in the following form:

$$U_{0,j}^{n+1/2} = g(0, z_j) \quad (27)$$

$$-a_{ij} U_{i-1,j}^{n+1/2} + c_{ij} U_{i,j}^{n+1/2} - b_{ij} U_{i+1,j}^{n+1/2} =$$

$$f_{i,j}, \quad i = 1, 2, \dots, M; j = 1, 2, \dots, N,$$

$$U_{M+1,j}^{n+1/2} = g(1, z_j).$$

Therefore, at each fixed j one has a linear system of M equations with a tridiagonal matrix. So, we can simply parallelize the computation in equation (27) by assigning a N/P system to each processor. For processor K^{th} , namely CPU(K-1), it will solve the system of equations for $j = (K-1)N/P + 1, \dots, KN/P$. With this parallelization strategy, the coefficient matrix needs to be distributed row-wise as demonstrated in Figure 4(a).

Equations of the second half step can be written as follows:

$$U_{i,0}^{n+1} = g(y_i, 0), \quad (28)$$

$$-a_{i,j} U_{i,j-1}^{n+1} + c_{i,j} U_{i,j}^{n+1} - b_{i,j} U_{i,j+1}^{n+1} =$$

$$f_{i,j}, \quad j = 1, 2, \dots, N; i = 1, 2, \dots, M,$$

$$U_{i,N+1}^{n+1} = g(y_i, 1).$$

There fore, at each fixed i one has a linear system of N equations with a tridiagonal matrix. Processor CPU(K-1) will solve the system of equations for $i = (K-1)M/P + 1, \dots, KM/P$. Figure 4(b) shows the column-wise direction of computation for the second step of the splitting scheme. We should compute in parallel on every strip in the direction of the x -axis for the first step and on every strip in the direction of the y -axis for the second step. Consequently, we need to redistribute the data between the row-splitting and column-splitting or vice versa.

Let us consider a grid of size $N \times M$ and assume that one node of the grid requires t seconds for computation. So, for all nodes of the grid there is required $(N \times M) \times t$ seconds. Let P be the number of processors and each processor works with $(N \times M) / P$ nodes of the

grid. So, each processor requires $((N \times M) / P) \times t$ seconds and $P \times (P - 1)$ times for updating the data blocks. Therefore, speedup is

$$S = \frac{(N \times M) \times t}{\frac{(N \times M) \times t}{P} + P \times (P - 1) \times \left(t_l + \frac{MessageSize}{Bandwidth} \right)} \quad (29)$$

The algorithm of the parallel domain decomposition is similar to the sequential algorithm but the components of $U_d, \langle \rho \rangle^{n+1}, \langle P_1 \rangle^{n+1}, \varepsilon^{n+1}, \langle v'w' \rangle^{n+1}, \langle u'^2 \rangle^{n+1}, \langle v'^2 \rangle^{n+1}, \langle w'^2 \rangle^{n+1}$, and Θ^{n+1} were computed by the domain decomposition algorithm of the splitting scheme while the components of V^*, W^* , and e^{n+1} were computed with parallelization over the number of equations.

Results and Discussion

We used Model 2 to implement the ideas of the functional and domain decompositions given above. First we utilized the idea of parallel functional decomposition. The analysis of the numerical model shows that it can be represented in the form of 5 independent groups of tasks. In the notations of Figure 3, we have $m_1 = 3, m_2 = 1, m_3 = 2, m_4 = 7$, and $m_5 = 1$. Jobs in each group can be executed independently in different processors. In the second approach we used the idea of parallel domain decomposition

to each job solved in the sequential algorithm. It means that the components of $U_d, \langle p_1 \rangle^{n+1}, \langle \rho \rangle^{n+1}, \varepsilon^{n+1}, \langle v'w' \rangle^{n+1}, \langle u'^2 \rangle^{n+1}, \langle v'^2 \rangle^{n+1}, \langle w'^2 \rangle^{n+1}$, and Θ^{n+1} were computed by the domain decomposition algorithm of the splitting scheme while the components of V^*, W^* , and e^{n+1} were computed by the explicit domain decomposition algorithm with all available processors. The performance results for the parallel computation runs for both clusters depend on the grid size. The numerical results of the parallel functional decomposition and parallel domain decomposition algorithms have been compared with the model using the sequential algorithm to guarantee the correctness of the parallel algorithms and are also compared with Lin and Pao's experimental data and Hassid's computational results (Lin and Pao, 1979; Hassid, 1980).

The comparison of results for the density Froude number $Fr = 31$ in the homogeneous ($g = 0$) and stratified fluids is presented in Figures 5 and 6. Figure 5(a) and Figure 6(a) show the dimensionless axial values of turbulent energy $e_0 = e_0(x) = e(x, 0, 0)$ in the case of drag and momentumless wakes, respectively. The symbols $\circ$ and $\bullet$ correspond to Lin and Pao's experimental data (Lin and Pao, 1979). The dashed lines correspond to the results of Hassid's numerical experiments (Hassid, 1980). The symbols ∇ and $\blacktriangledown$ indicate the computational results by the sequential algorithm. The symbols

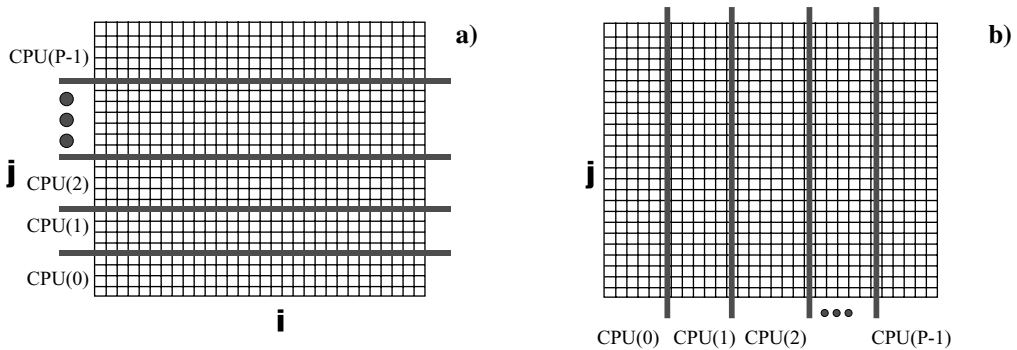


Figure 4. The sketch of the domain decomposition process in the direction of the y-axis and x-axis

▷ and ► indicate the computational results by the domain decomposition algorithm. The symbols □ and ■ indicate the computational results by the functional decomposition algorithm. Analogous data for the time variation of the dimensionless axial velocity defect $U_d = U_d(x, 0, 0)$ are presented in Figure 5(b) and Figure 6(b) for drag and momentumless wakes, respectively.

The dynamics of a turbulent wake and internal waves generated by the wake in a linearly stratified fluid is illustrated in Figure 7. The density Froude number was assumed to be

equal 280 which corresponds to the conditions of one of the laboratory experiments of Lin and Pao (1979). Figure 7 shows isolines $\langle \rho_1 \rangle / (aD \rho_0) = \text{const}$ in drag wake for the moments of timer $t/T = 2$ and $t/T = 3$ respectively. The level of isolines varies from -0.015 to 0.015 with a stepping of 0.005. It is not surprise that patterns of isolines drawing by the data from functional and domain decomposition algorithms are the same. All the considered algorithms yield a satisfactory agreement with the experimental and numerical data. The other main characteristics of the wake (the horizontal and vertical size of

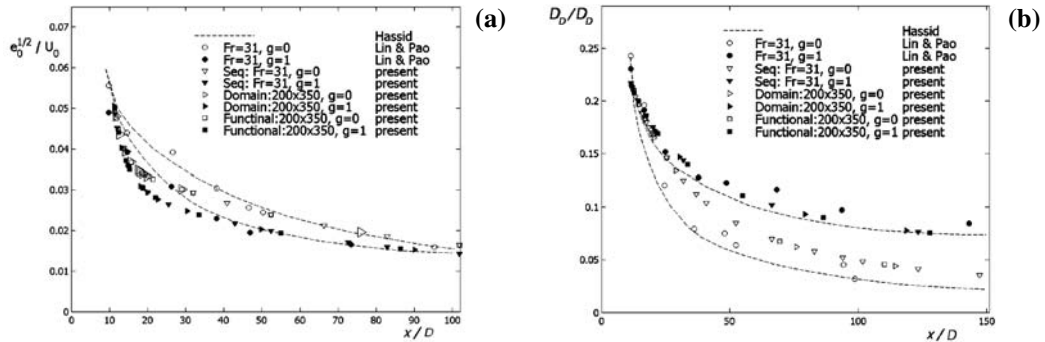


Figure 5. Comparison of the axial values of the turbulent energy $e_0(x)$ (a), and the longitudinal velocity component defect, $U_d(x)$ (b), in the wake behind a towed body calculated by sequential and parallel algorithms with Lin and Pao's (1979) experimental data and Hassid's (1980) computational results

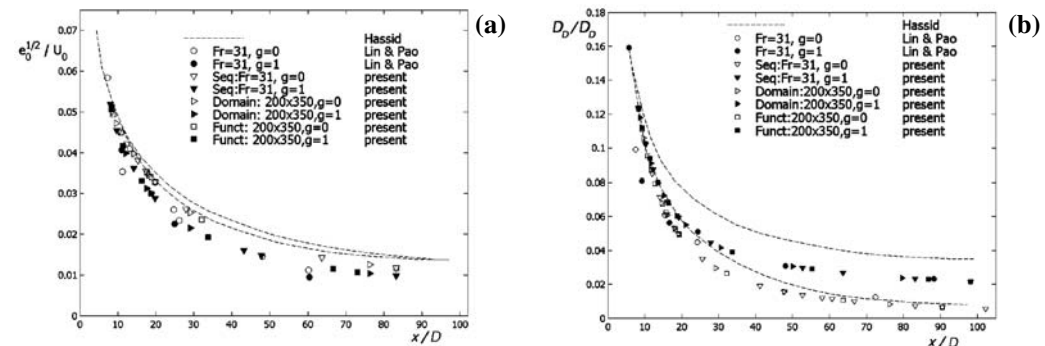


Figure 6. Comparison of the axial values of the turbulent energy, $e_0(x)$ (a), and the longitudinal velocity component defect, $U_d(x)$ (b), in the wake behind a self-propelled body calculated by sequential and parallel algorithms with Lin and Pao's (1979) experimental data and Hassid's (1980) computational results

but not by the number of processors. The overall simulation time required to compute the wake characteristics up to time $t/T = 20$ by the parallel domain decomposition is much less than the parallel functional decomposition. For instance, the simulation time using seven processors (and a grid of $600 \times 1,500$ nodes) for the domain decomposition method is 2851.70 sec, while the functional decomposition method would need 4927.48 sec. The efficiency and details of computational time for both algorithms are demonstrated in Tables 1 and 2. The data are

shown for SUT-HPCC where speedup and efficiency higher compared with Cluster I.

Figure 9 shows the comparison between the theoretical and numerical speedup results of the parallel functional and domain decomposition algorithms, respectively. The results are for the grid with $800 \times 1,550$ nodes, and the numerical computation were performed on SUT-HPCC. Since we knew the sequential run time for all subroutines, the latency time, bandwidth, and message size of data, so we estimated the theoretical speedup by equations

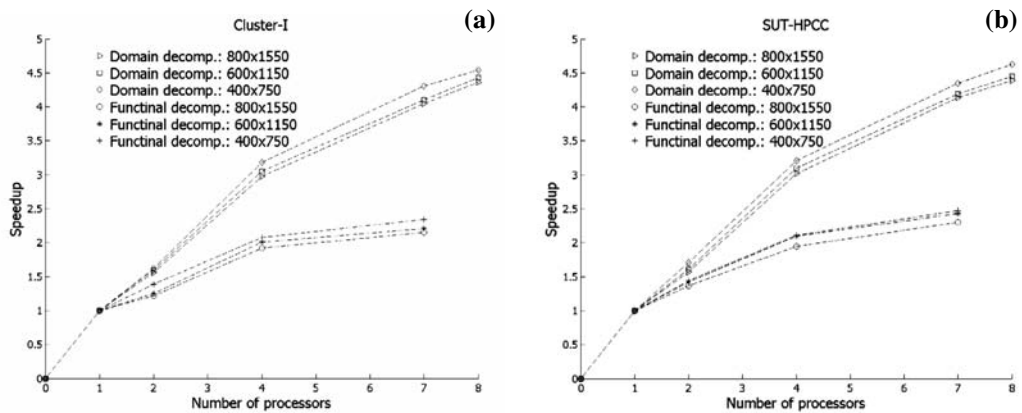


Figure 8. Speedup results on Cluster-I, (a), and SUT-HPCC (b)

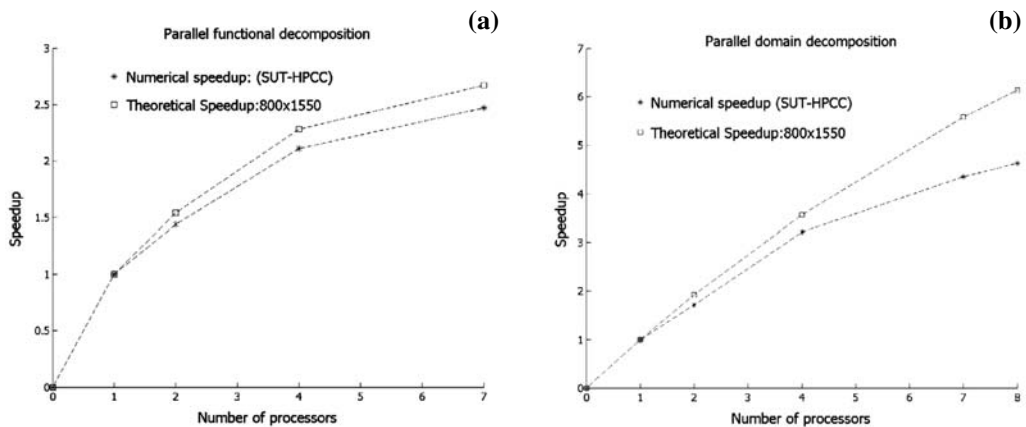


Figure 9. Comparison between theoretical and numerical speedup results of the functional (a), and domain decomposition (b) on SUT-HPCC

(25) and (29) for the parallel functional and domain decomposition algorithms, respectively. We can see that the numerical speedups of both parallel decomposition algorithms are lower than the theoretical speedups. In reality, an additional overhead must be paid for communication and synchronization. This overhead changes the speedup. It is worth noting, that equation (25) predicts speedup for the functional decomposition method more accurate by compared with equation (29), which predicts speedup for the domain decomposition method.

Conclusions

We have developed the parallel algorithms based on the functional and domain decompositions techniques. The developed parallel algorithms have been successfully applied to the numerical simulation of turbulent wake dynamics behind towed and self-propelled bodies in a linear

stratified fluid. The following conclusions can be drawn from the present work:

- The results of the theoretical estimates and numerical experiments demonstrate the significant speedup of both parallel algorithms compared with the sequential one. The results of the numerical simulations of turbulent wake dynamics in a stratified fluid using parallel algorithms are in good agreement with the experimental data and the data of the numerical simulation by the sequential code.
- The run time of the functional decomposition algorithm is about two times faster than the run time of the sequential code, while the run time of the domain decomposition algorithm is more than 4 times faster than the sequential code when seven processors are used.
- To distribute the tasks between the proces-

Table 1. Speedup, efficiency and computational time. Results of functional decomposition algorithm on SUT-HPCC

Grid size		600 × 1,150			800 × 1,550		
NCPUs	Time	Speedup	Efficiency	Time	Speedup	Efficiency	
1	11,952.72			25,420.44			
2	8,417.48	1.42	0.7100	17,614.25	1.44	0.7200	
4	5,683.31	2.10	0.5250	122,058.83	2.11	0.5275	
7	4,927.48	2.43	0.3471	10,291.68	2.47	0.3529	

Table 2. Speedup, efficiency and computational time. Results of parallel domain decomposition algorithm on SUT-HPCC

Grid size		600 × 1,150			800 × 1,550		
NCPUs	Time	Speedup	Efficiency	Time	Speedup	Efficiency	
1	11,952.72			25,420.44			
2	7,431.28	1.61	0.8050	14,869.83	1.71	0.8550	
4	3,850.87	3.10	0.7750	7,916.05	3.21	0.8025	
7	2,851.70	4.19	0.5986	5,846.71	4.35	0.6214	
8	2,686.21	4.45	0.5563	5,487.65	4.63	0.5788	

sors and organized communications, the parallel functional decomposition algorithm is simpler when compared with the parallel domain decomposition algorithm. But the functional decomposition algorithm has a limitation on the number of processors used and has a lower limit of maximal speedup.

Finally, the present research shows that both developed parallel algorithms are viable tools for numerical simulations of turbulent wake dynamics in stratified fluid and can serve as a basis for numerical experiments with more complicated models of turbulence.

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