

OSCILLATION CRITERIA OF CERTAIN FOURTH ORDER DIFFERENCE EQUATIONS WITH DAMPING TERM

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Abstract

The fourth-order delay difference equation's oscillation condition is the subject of this paper. The basic concept is to use comparison methods to get the oscillation result and Property B. Some previous findings are expanded upon and improved by our research. To illustrate the key findings, examples are provided.

Keywords: Damping terms; Delay equation; Fourth order; Oscillatory behaviour

Introduction

Many researchers have been concentrated on finding the oscillatory solutions for difference equations. Even though third-order difference equations are found in physics, biology, and other mathematical disciplines, they have not gotten enough investigation (Elaydi, 1996; Agarwal, 2000; Kelley & Peterson, 2001; Grace, 2005; Kaleeswari & Selvaraj, 2016a, 2016b; Chatzarakis *et al.*, 2022). The study of the oscillatory properties of third and higher-order equations has attracted a lot of attention in recent years. This is because issues in economics, mathematical biology, and many other areas of mathematics commonly involve the use of such an equation (Artzrouni, 1985; Kaleeswari *et al.*, 2014; Selvaraj & Kaleeswari, 2015; Kaleeswari, 2020).

Recently, Selvaraj and Kaleeswari (2013) discussed the oscillation criteria using comparison method. Throughout this article, researchers examine the oscillatory behaviour of a fourth-order

non-linear delay difference equation with positive damping components of the type and discuss about the solutions are either oscillatory or having Property (B). Here $\{a_n\}, \{b_n\}$ are the sequences of positive real numbers, n_0 is a non-negative number. We'll make the following assumptions in the continuation and discuss about the solutions are either oscillatory or having Property (B). Here $\{a_n\}, \{b_n\}$ are the sequences of positive real numbers, n_0 is a non-negative number. We'll make the following assumptions in the continuation:

$$\Delta(a_n \Delta(b_n \Delta(x_n)^\eta)) + p_n \Delta(x_{n+1})^\eta + q_n f(x_{n-\sigma}) = 0, n \geq n_0, \quad (1.1)$$

(A1) p_n, q_n are positive series $\forall n \geq n_0$;

(A2) σ is a positive number and η is a ratio of odd positive numbers;

(A3) $xf(x) > 0$, f is increasing function for $x \neq 0$, and $-f(-pq) \geq f(pq) \geq f(p)f(q)$ for $p, q > 0$.

A non-trivial real sequence $\{x_n\}$ that fulfils equation (1.1) $\forall n \geq n_0$ is referred to as a solution of equation (1.1). If the answer to equation (1.1) is neither eventually positive nor eventually negative, it is said to be oscillatory; otherwise, it is said to be non-oscillatory.

The following is how the paper is set up: In Section 2, we provide some basic elaborations that will help to demonstrate our main results. In Section 3, it is given that, equation (4.2) has Property B. Section 4, demonstrates the oscillation theorems. Finally, examples are provided to show the applicability of the findings.

Preliminary Results

This section will describe about the behaviour of solutions of equation (1.1) and the difference equation,

$$\Delta(a_n \Delta(y_n)) + \frac{p_n}{b_{n+1}} y_{n+1} = 0, n \geq n_0. \quad (2.1)$$

The following Theorem recall a number of intriguing and significant findings regarding equation (1.1)

Theorem 2.1. Suppose $\{y_n\}$ is a positive solution of equation (2.1). Then

$$L(x_n) = \Delta(a_n \Delta(b_n \Delta(\Delta x_n)^\eta)) + p_n \Delta(\Delta x_{n+1})^\eta$$

must be written by

$$L(x_n) = \frac{1}{y_{n+1}} \Delta \left(a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) \right), \quad (2.2)$$

Proof. From (2.1) we have,

$$p_n = -\frac{b_{n+1}}{y_{n+1}} \Delta(a_n \Delta(y_n)).$$

It is easy to observe that

$$\begin{aligned} L(x_n) &= \frac{1}{y_{n+1}} \left[\Delta \left(a_n y_n y_{n+1} \left[\frac{\Delta(b_n \Delta(\Delta x_n)^\eta) y_{n+1}}{y_n y_{n+1}} \right] \right) \right. \\ &\quad \left. - \left[\frac{b_{n+1} \Delta(\Delta x_{n+1})^\eta \Delta y_n}{y_n y_{n+1}} \right] \right] \\ &= \frac{1}{y_{n+1}} \left[\Delta(a_n \Delta(b_n \Delta(\Delta x_n)^\eta) y_{n+1}) \right. \\ &\quad \left. - \Delta(a_n b_{n+1} \Delta(\Delta x_{n+1})^\eta \Delta y_n) \right] \\ &= \frac{1}{y_{n+1}} \Delta y_{n+1} a_{n+1} \Delta(b_{n+1} \Delta(x_{n+1})^\eta) \\ &\quad + \frac{1}{y_{n+1}} y_{n+1} \Delta(a_n \Delta(b_n \Delta(\Delta x_n)^\eta)) \\ &\quad - \frac{1}{y_{n+1}} \Delta y_{n+1} a_{n+1} \Delta(b_{n+1} \Delta(x_{n+1})^\eta) \\ &\quad - \frac{b_{n+1}}{y_{n+1}} \Delta(\Delta x_{n+1})^\eta \Delta(a_n \Delta y_n) \end{aligned}$$

$$L(x_n) = \Delta(a_n \Delta(b_n \Delta(\Delta x_n)^\eta)) + p_n \Delta(\Delta x_{n+1})^\eta. \quad (2.3)$$

Hence, the proof is completed.

Corollary 2.2. If equation (2.1) has a positive solution $\{y_n\}$, may be expressed as follows:

$$\Delta \left(a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) \right) + q_n y_{n+1} f(x_{n-\sigma}) = 0. \quad (2.4)$$

Proof. The proof is direct impact of Theorem 2.1.

Lemma 2.3. (Agarwal *et al.*, 2005) If

$$a_n \geq 1, \sum_{n=n_0}^{\infty} \frac{p_n}{b_{n+1}} < \infty, \limsup_{n \rightarrow \infty} \sum_{s=n}^{\infty} \frac{p_s}{b_{s+1}} < \frac{1}{4}, \quad (2.5)$$

then the solution of (2.1) is a positive solution.

Lemma 2.4. (Agarwal *et al.*, 2005) Assume the condition (2.5) holds. Then $\exists$ a non-oscillatory solution to the equation (2.1) fulfilling

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n y_n y_{n+1}} = \infty. \quad (2.6)$$

Now let us assume that

$$\sum_{n=n_0}^{\infty} \left(\frac{y_n}{b_n} \right)^{\frac{1}{\eta}} = \infty \quad (2.7)$$

where $\{y_n\}$ is a positive solution which exists by Lemma 2.3.

Lemma 2.5. (Györi & Ladas, 1991) Let $\{q_n\}$ be a positive sequence, σ is a positive integer and η is a ratio of odd positive integers. If the difference inequality has an eventually positive solution, then the difference equation $\Delta u_n + q_n u_{n-\sigma}^\eta = 0$ also has an eventually positive solution.

$$\Delta u_n + q_n u_{n-\sigma}^\eta \leq 0$$

Next, we study the behaviour of solution of equation (2.4) which is equivalent to the equation (1.1).

Assume the system (2.1) has a positive solution $\{y_n\}$ that fulfils (2.6) and let the condition (2.5) hold. If condition (2.7) is fulfilled, then the positive solution of system (2.4) has the following two cases:

$$\begin{aligned} (S1) \quad & (\Delta x_n)^\eta < 0, \Delta(\Delta x_n)^\eta > 0, \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) \\ & < 0, \Delta \left(a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) \right) < 0, \end{aligned}$$

(or)

$$\begin{aligned} (S2) \quad & (\Delta x_n)^\eta > 0, \Delta(\Delta x_n)^\eta > 0, \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) > 0 \\ & \Delta \left(a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) \right) < 0. \end{aligned}$$

Suppose the condition (2.7) is not satisfied, then other type of non-oscillatory is possible.

$$\begin{aligned} (S3) \quad & (\Delta x_n)^\eta < 0, \Delta(\Delta x_n)^\eta < 0, \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) \\ & < 0, \Delta \left(a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) \right) \\ & < 0. \end{aligned}$$

In the following section, we discuss about Property (B).

Property B

We provide certain criteria in this section for equation (1.1) have Property B.

Definition 3.1. Every non-oscillatory solution to equation (1.1) has Property B if it meets the condition $\{x_n\}$

$$x_n \Delta x_n < 0. \quad (3.1)$$

In the following theorem we describe equation (1.1) has the Property B condition its proved by using contradiction.

Theorem 3.2. Let the solution of the equation (2.1) be positive and meet conditions (2.6) and (2.7) and assume (2.5). If the equation

$$\begin{aligned} \Delta u_n + q_n y_{n+1} f \left(\sum_{s=n_1}^{n-\sigma-1} \left(\sum_{t=s_1}^{s-1} \frac{y_t}{b_t} \sum_{i=t_1}^{t-1} \frac{1}{a_i y_i y_{i+1}} \right)^{\frac{1}{\eta}} \right. \\ \left. f \left(u_{n-\sigma}^{\frac{1}{\eta}} \right) \right) = 0, n \geq n_1 \end{aligned} \quad (3.2)$$

is oscillatory, then (1.1) has Property B.

Proof. Suppose $\{x_n\}$ is a positive solution of equation (1.1) and hence of (2.4). It follows that $\{x_n\}$ belongs to either S1 or S2. If $x_n \in S2$, then

$$u_n = a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta(\Delta x_n)^\eta \right) > 0$$

and $\{u_n\}$ is decreasing. Immediately we can write

$$\begin{aligned} \frac{b_n}{y_n} \Delta(\Delta x_n)^\eta &\geq \sum_{s=n_1}^{n-1} \frac{1}{a_s y_s y_{s+1}} a_s y_s y_{s+1} \Delta \left(\frac{b_s}{y_s} \Delta(\Delta x_s)^\eta \right) \\ &\geq u_n \sum_{s=n_1}^{n-1} \frac{1}{a_s y_s y_{s+1}} \end{aligned}$$

The aforementioned inequality from n_1 to $n-1$ is summing to give

$$(\Delta x_n)^\eta \geq \sum_{s=n_1}^{n-1} \frac{y_s u_s}{b_s} \sum_{t=s_1}^{s-1} \frac{1}{a_t y_t y_{t+1}}$$

The aforementioned inequality from n_1 to $n-1$ is once more summing up, giving us

$$x_n \geq \sum_{s=n_1}^{n-1} \left(\sum_{t=s_1}^{s-1} \frac{y_t u_t}{b_t} \sum_{i=t_1}^{t-1} \frac{1}{a_i y_i y_{i+1}} \right)^{\frac{1}{\eta}}$$

Thus,

$$x_{n-\sigma} \geq \sum_{s=n_1}^{n-\sigma-1} \left(\sum_{t=s_1}^{s-1} \frac{y_t}{b_t} \sum_{i=t_1}^{t-1} \frac{1}{a_i y_i y_{i+1}} \right)^{\frac{1}{\eta}}$$

Using the aforementioned inequality in (2.4) and by (A3), we obtain

$$\begin{aligned} & -\Delta u_n \\ & \geq q_n y_{n+1} f \left(u_{n-\sigma}^{\frac{1}{\eta}} \sum_{s=n_1}^{n-\sigma-1} \left(\sum_{t=s_1}^{s-1} \frac{y_t}{b_t} \sum_{i=t_1}^{t-1} \frac{1}{a_i y_i y_{i+1}} \right)^{\frac{1}{\eta}} \right) \\ & \geq q_n y_{n+1} f \left(\sum_{s=n_1}^{n-\sigma-1} \left(\sum_{t=s_1}^{s-1} \frac{y_t}{b_t} \sum_{i=t_1}^{t-1} \frac{1}{a_i y_i y_{i+1}} \right)^{\frac{1}{\eta}} \right) \\ & * f \left(u_{n-\sigma}^{\frac{1}{\eta}} \right) \end{aligned}$$

Therefore $\{u_n\}$ is a positive solution of

$$\begin{aligned} & \Delta u_n \\ & + q_n y_{n+1} f \left(\sum_{s=n_1}^{n-\sigma-1} \left(\sum_{t=s_1}^{s-1} \frac{y_t}{b_t} \sum_{i=t_1}^{t-1} \frac{1}{a_i y_i y_{i+1}} \right)^{\frac{1}{\eta}} \right) f \left(u_{n-\sigma}^{\frac{1}{\eta}} \right) \\ & \leq 0 \end{aligned}$$

which contradicts by Lemma 2.5, so the equation (3.2) is oscillatory. Hence equation (1.1) has Property B

Corollary 3.3. Let hypothesis of Lemma 2.3 hold, $f(u)=u^\eta$ and assume that the solution of the equation (2.1) is positive and meet conditions (2.6) and (2.7). If $\lim_{n \rightarrow \infty} \inf$

$$\begin{aligned} & \sum_{v=n_0-\sigma}^{n-\sigma-1} q_v y_{v+1} \left(\sum_{s=v_1}^{v-\sigma-1} \left(\sum_{t=s_1}^{s-1} \frac{y_t}{b_t} \sum_{j=t_1}^{t-1} \frac{1}{a_j y_j y_{j+1}} \right)^{\frac{1}{\eta}} \right) \\ & > \left(\frac{\sigma}{\sigma+1} \right)^{\sigma+1} \end{aligned} \quad (3.3)$$

then system (1.1) has Property (B).

The result of the subsequent theorem is an asymptotic solution to the equation (1.1), which meets the Property B condition.

Theorem 3.4. Suppose the condition (2.7) be hold and let (1.1) posses Property (B). If

$$\lim_{n \rightarrow \infty} \sum_{s=n_1}^{n-1} \left(\sum_{t=s}^{\infty} \frac{y_t}{b_t} \sum_{i=t}^{\infty} \frac{1}{a_i y_i y_{i+1}} \sum_{j=1}^{\infty} q_j y_{j+1} \right)^{\frac{1}{\eta}} = \infty \quad (3.4)$$

then every non-oscillatory solution of system (1.1) tends to 0 as $n \rightarrow \infty$.

Proof. Assume that $\{x_n\}$ is a non-oscillatory solution of (1.1) and hence of (2.4) such that $x_n > 0$. Also assume that $x_n \Delta x_n < 0$ for all $n \geq n_0$. Then there exists a non-negative integer $w \geq 0$, such that $\lim_{n \rightarrow \infty} x_n = w$. Now contradictory assume that $w > 0$. Using (A3), (2.4) can be written as

$$\begin{aligned} & -\Delta \left(a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta (\Delta x_n)^\eta \right) \right) = q_n y_{n+1} f(x_{n-\sigma}) \\ & \geq q_n y_{n+1} f(w). \end{aligned}$$

The aforementioned inequality is summing up from n to ∞ , and we carry

$$-\Delta \left(\frac{b_n}{y_n} \Delta (\Delta x_n)^\eta \right) \geq \frac{1}{a_n y_n y_{n+1}} f(w) \sum_{s=n}^{\infty} q_s y_{s+1}.$$

Summing up the resulting inequality from n to ∞ implies

$$-\Delta (\Delta x_n)^\eta \geq \frac{y_n}{b_n} f(w) \sum_{s=n}^{\infty} \frac{1}{a_s y_s y_{s+1}} \sum_{t=s}^{\infty} q_t y_{t+1}$$

Again summing, we have

$$-(\Delta x_n)^\eta \geq f(w) \sum_{s=n}^{\infty} \frac{y_s}{b_s} \sum_{t=s}^{\infty} \frac{1}{a_t y_t y_{t+1}} \sum_{i=t}^{\infty} q_i y_{i+1}$$

Repeating the summation from n_1 to $n-1$, we obtain

$$\begin{aligned} & x_n \\ & \leq x_{n_1} \\ & - f(w) \sum_{s=n_1}^{n-1} \left(\sum_{t=s}^{\infty} \frac{y_t}{b_t} \sum_{i=t}^{\infty} \frac{1}{a_i y_i y_{i+1}} \sum_{j=1}^{\infty} q_j y_{j+1} \right)^{\frac{1}{\eta}}. \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and from equation (3.4), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n &\leq x_{n_1} \\ &- f(w) \lim_{n \rightarrow \infty} \sum_{s=n_1}^{n-1} \left(\sum_{t=s}^{\infty} \frac{y_t}{b_t} \sum_{i=t}^{\infty} \frac{1}{a_i y_i y_{i+1}} \sum_{j=i}^{\infty} q_j y_{j+1} \right)^{\frac{1}{\eta}} \\ &= -\infty \end{aligned}$$

which contradicts. Therefore $w = 0$ and hence every non-oscillatory solution of (1.1) tends to 0 as $n \rightarrow \infty$

In the next section we discuss the oscillatory criteria of the fourth order difference equation.

Oscillation Criteria

In Theorem 4.1, we assume the solution of (1.1) is positive and it belongs to the condition S1 proceed the proof by summing by parts method, result is contradiction to our statement so solution of the equation (1.1) is oscillatory

Theorem 4.1. Suppose $\{y_n\}$ is a positive solution of system (2.1) meet conditions (2.6) and (2.7). Let (2.5) be true. Suppose that there is a positive integer k with $k < \frac{\sigma}{2}$ such that (3.2) and

$$\begin{aligned} \Delta x_n &+ \left(\sum_{s=n}^{n+k} \frac{y_s}{b_s} \sum_{t=s}^{s+k} \frac{1}{a_t y_t y_{t+1}} \sum_{i=t}^{t+k} q_i y_{i+1} \right)^{\frac{1}{\eta}} f^{\frac{1}{\eta}}(x_{n-\sigma+3k}) \\ &= 0 \end{aligned} \quad (4.1)$$

are both oscillatory, then equation (1.1) is oscillatory.

Proof. Assume $\{x_n\}$ is an eventually positive solution of (1.1). Consequently $\{x_n\}$ belongs to either (S1) or (S2). The oscillation of equation (3.2) suggests that (S2) is empty according to the Theorem 3.2. Hence $\{x_n\}$ belongs to S1.

Taking summation of equation (2.4) from n to $n+k$, gives

$$-a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} (\Delta x_n)^\eta \right) \geq \sum_{s=n}^{n+k} q_s y_{s+1} f(x_{s-\sigma}).$$

It implies

$$-\Delta \left(\frac{b_n}{y_n} \Delta (\Delta x_n)^\eta \right) \geq \frac{f(x_{n-\sigma+k})}{a_n y_n y_{n+1}} \sum_{s=n}^{n+k} q_s y_{s+1}$$

Summing the final inequality from n to $n+k$ once again, we obtain $-\Delta (\Delta x_n)^\eta \geq \frac{y_n}{b_n} f(x_{n-\sigma+2k}) \sum_{s=n}^{n+k} \frac{1}{a_s y_s y_{s+1}} \sum_{t=s}^{s+k} q_t y_{t+1}$ Summing the aforementioned inequality from n to $n+k$ once more, we obtain

$$\begin{aligned} & -(\Delta x_n)^\eta \\ & \geq f(x_{n-\sigma+3k}) \sum_{s=n}^{n+k} \frac{y_s}{b_s} \sum_{t=s}^{s+k} \frac{1}{a_t y_t y_{t+1}} \sum_{i=t}^{t+k} q_i y_{i+1} \end{aligned}$$

This implies

$$\begin{aligned} & -\Delta x_n \\ & \geq \left(\sum_{s=n}^{n+k} \frac{y_s}{b_s} \sum_{t=s}^{s+k} \frac{1}{a_t y_t y_{t+1}} \sum_{i=t}^{t+k} q_i y_{i+1} \right)^{\frac{1}{\eta}} f^{\frac{1}{\eta}}(x_{n-\sigma+3k}) \\ & \Delta x_n \\ & + \left(\sum_{s=n}^{n+k} \frac{y_s}{b_s} \sum_{t=s}^{s+k} \frac{1}{a_t y_t y_{t+1}} \sum_{i=t}^{t+k} q_i y_{i+1} \right)^{\frac{1}{\eta}} f^{\frac{1}{\eta}}(x_{n-\sigma+3k}) \\ & \leq 0 \end{aligned}$$

The difference equation (4.1) has a positive solution according to Lemma 2.5, which is in contradiction.

As a result, every oscillatory solution $\{x_n\}$ to the equation (1.1) exists. Hence, the proof is completed.

Corollary 4.2. Suppose $\{y_n\}$ is a positive solution of (2.1) meet conditions (2.6) and (2.7). Let (2.5) be true. Suppose that the condition (3.3) retains and there is a positive integer k with $k < \frac{\sigma}{2}$ such that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sum_{s=n_0-\sigma+3k}^{n-1} \left(\sum_{t=s}^{s+k} \frac{y_t}{b_t} \sum_{i=t}^{t+k} \frac{1}{a_i y_i y_{i+1}} \sum_{j=i}^{i+k} q_j y_{j+1} \right)^{\frac{1}{\eta}} \\ > \left(\frac{\sigma-3k}{\sigma-3k+1} \right)^{\sigma-3k+1}. \end{aligned} \quad (4.2)$$

Then equation (1.1) is oscillatory.

Proof. As a result of Theorem 3.2 and Theorem 4.1, we can see that conditions (3.3) and (4.2) indicate that equations (3.2) and (4.1) are oscillatory.

In Theorem 4.1, we assume the solution of (1.1) is positive and it belongs to the condition S3 proceed the proof by summing by parts method,

result is contradiction to our statement so solution of the equation (1.1) is oscillatory

Theorem 4.3. Suppose $\{y_n\}$ is a positive solution of system (2.1) meet conditions (2.6) and (2.7). Let (2.5) be true. If there is a positive integer k with $k < \frac{\sigma}{2}$ such that (3.2) and (4.1) are oscillatory and if

$$\sum_{s=n_1}^{n-1} \left(\sum_{t=n_1}^{s-1} \frac{y_t}{b_t} \sum_{i=n_1}^{t-1} \frac{1}{a_i y_i y_{i+1}} \right)^{\frac{1}{\eta}} * \sum_{s=n_1}^{n-1} \left(\sum_{j=n_1}^{i-1} q_j y_{j+1} f(x_{j-\sigma}) \right)^{\frac{1}{\eta}} = \infty. \quad (4.3)$$

then (1.1) is oscillatory.

Proof. Consider $\{x_n\}$ a non-oscillatory solution of system (1.1) and hence (2.4) such that $x_n > 0$. From Theorem 3.2 and Theorem 4.1, the conditions (S1) and (S2) are empty. Assume that $\{x_n\} \in S3$. Now summing (2.4) from n_1 to $n-1$ becomes,

$$\begin{aligned} & a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta((\Delta x_n)^\eta) \right) \\ &= - \sum_{s=n_1}^{n-1} q_s y_{s+1} f(x_{s-\sigma}) \\ &+ a_{n_1} y_{n_1} y_{n_1+1} \Delta \left(\frac{b_{n_1}}{y_{n_1}} \Delta((\Delta x_{n_1})^\eta) \right). \\ & a_n y_n y_{n+1} \Delta \left(\frac{b_n}{y_n} \Delta((\Delta x_n)^\eta) \right) \\ &\leq - \sum_{s=n_1}^{n-1} q_s y_{s+1} f(x_{s-\sigma}), \quad n \geq n_1. \quad (4.4) \end{aligned}$$

The aforementioned inequality from n_1 to $n-1$ is summing to give us

$$\frac{b_n}{y_n} \Delta((\Delta x_n)^\eta) \leq - \sum_{s=n_1}^{n-1} \frac{1}{a_s y_s y_{s+1}} \sum_{t=s_1}^{s-1} q_t y_{t+1} f(x_{t-\sigma})$$

The result of summing the aforementioned inequality from n_1 to $n-1$ is

$$(\Delta x_n)^\eta \leq - \sum_{s=n}^{n-1} \frac{y_s}{b_s} \sum_{t=s_1}^{s-1} \frac{1}{a_t y_t y_{t+1}} \sum_{i=t_1}^{t-1} q_i y_{i+1} f(x_{i-\sigma})$$

Again summing, we obtain

$$x_n \leq - \sum_{s=n}^{n-1} \left[\sum_{t=s}^{s-1} \frac{y_t}{b_t} \sum_{i=t_1}^{t-1} \frac{1}{a_i y_i y_{i+1}} \sum_{j=i_1}^{i-1} q_j y_{j+1} f(x_{j-\sigma}) \right]^{\frac{1}{\eta}}$$

which contradicts positivity of x_n by (4.3). This completes the proof.

Examples

Based the previous finding results we discuss examples. In Example 5.1, set the parameter values according to the result of the Property B theorems and solution of the equation (1.1) is asymptotic.

Example 5.1. Take the delay difference equation

$$\begin{aligned} & \Delta \left(\frac{1}{n+1} \Delta \left((n+1) \Delta(\Delta x_n)^{\frac{1}{5}} \right) \right) \\ &+ \frac{1}{(n+1)(n+2)} \Delta(\Delta x_{n+1})^{\frac{1}{5}} \\ &+ (n+1) x_{n-1}^{\frac{1}{5}} = 0 \quad (5.1) \end{aligned}$$

Here, $a_n = \frac{1}{n+1}$, $b_n = (n+1)$, $p_n = \frac{1}{(n+1)(n+2)}$, $q_n = (n+1)$, $\eta = \frac{1}{5}$ and $\sigma = 1$. Put these values in (2.1), we get

$$\Delta \left(\frac{1}{n+1} \Delta y_n \right) + \frac{1}{(n+1)(n+2)^2} y_{n+1} = 0 \quad (5.2)$$

From Lemma 2.3, system (5.2) has a non-oscillatory solution. Infact, $\{y_n\} = (n+1)$ is a nonoscillatory solution of (5.2) that satisfies the conditions

$$\sum_{n=n_0}^{\infty} \frac{1}{(n+2)} = \infty$$

and

$$\sum_{n=n_0}^{\infty} 1 = \infty$$

From corollary 3.3, equation (3.3) gives

$$\liminf_{n \rightarrow \infty} \sum_{v=n_0-\sigma}^{n-1} (v+1)(v+2) \left(\sum_{s=v_1}^{v-2} \left(\sum_{t=s_1}^{s-1} \sum_{j=t_1}^{t-1} \frac{1}{(j+2)} \right)^{\frac{1}{5}} \right) > \frac{1}{4}$$

so (5.1) has Property (B). Also we can see

$$\lim_{n \rightarrow \infty} \sum_{n_1}^{n-1} \left(\sum_{t=s}^{\infty} \sum_{i=t}^{\infty} \frac{1}{(i+2)} \sum_{j=i}^{\infty} (n+1)(n+2) \right) = \infty$$

Therefore, according to Theorem 3.4, every non-oscillatory solution of (5.1) tends to 0 as $n \rightarrow \infty$.

In Example 5.2, set the parameter values according to the result of the oscillatory theorems and solution of the equation (1.1) is oscillatory.

Example 5.2. Take the delay difference equation

$$\Delta \left(\frac{1}{n+1} \Delta((n+1)\Delta(x_n)^5) \right) + \frac{1}{(n+1)(n+2)} \Delta(\Delta x_{n+1})^5 + (n+4)x_{n-1}^5 = 0 \quad (5.3)$$

Here, $a_n = \frac{1}{n+1}$, $b_n = (n+1)$, $p_n = \frac{1}{(n+1)(n+2)}$, $q_n = n+4$, $\eta = 5$ and $\sigma = 1$. Put these values in (2.1), we get

$$\Delta \left(\frac{1}{n+1} \Delta y_n \right) + \frac{1}{(n+1)(n+2)^2} y_{n+1} = 0 \quad (5.4)$$

Then Equation (5.4) has nonoscillatory solution $\{y_n\} = (n+1)$ which satisfies the conditions

$$\sum_{n=n_0}^{\infty} \frac{1}{(n+2)} = \infty$$

and

$$\sum_{n=n_0}^{\infty} 1 = \infty$$

If we assume $k=1$, then (4.2) becomes

$$\liminf_{n \rightarrow \infty} \sum_{s=n_0-1}^{n-1} \left(\sum_{t=s}^{s+k} t + 2 \sum_{i=t}^{t+k} \frac{1}{(i+2)} \sum_{j=i}^{i+k} (j+4)(j+2) \right)^{\frac{1}{5}} > \frac{1}{2}$$

and (3.3) also holds. So by corollary 4.2 every solution of (5.3) is oscillatory.

Conclusion

The findings in this study are highly general, enhance and complete earlier findings for certain cases of equation (1.1). Furthermore, the monotonic characteristics of the solutions which are described in the theorems are proved by using the comparison principle and summation averaging methodology etc. Also the examples determine the main results.

Declaration

Ethical Approval: Not applicable, as the study did not involve human or animal participants.

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