

INVESTIGATION OF MULTIVALENT FUNCTIONS IN COMPLEX HILBERT SPACE EMPLOYING FRACTIONAL CALCULUS AND BOREL DISTRIBUTION SERIES

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Abstract

The purpose of the present paper is to determine the necessary and sufficient condition for the power series $M_r(\rho, B)h(B)$ whose coefficients are probabilities of the Borel distribution to be in the subclasses $TS_r^*(\gamma, \delta, \eta, B)$ and $TK_r(\gamma, \delta, \eta, B)$ of multivalent functions defined in the open unit disk. By introducing application of fractional calculus, we derive certain interesting geometric properties such as coefficient estimates, extreme points and convex combination.

Keywords: Analytic Functions; Multivalent Functions; Borel Distribution; Fractional Calculus

Introduction

Let $S(r, m)$ represent the class of functions that are analytic and multivalent in the open unit disk $\mathcal{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with the Taylor series expansion

$$h(\zeta) = \zeta^r + \sum_{k=r+m}^{\infty} b_k \zeta^k \quad (r, m \in \mathbb{N}). \quad (1.1)$$

We denote $T(r, m) \subset S(r, m)$ consisting of functions of the form

$$h(\zeta) = \zeta^r - \sum_{k=r+m}^{\infty} b_k \zeta^k \quad (b_k \geq 0, r, m \in \mathbb{N}). \quad (1.2)$$

Let $\mathcal{H}$ denote a Complex Hilbert space and B be a bounded linear operator on $\mathcal{H}$. For a function h that is analytic on the unit disk $\mathcal{U}$, the operator $h(B)$ is defined by the Riez- Dunford integral (Dunford & Schwartz, 1958) as

$$h(B) = \frac{1}{2\pi i} \int_C h(\zeta)(\zeta I - B)^{-1}$$

Where I is the identity operator on $\mathcal{H}$ and C is a positively oriented simple closed rectifiable contour contained within $\mathcal{U}$, enclosing the spectrum $s(B)$. This operator can also be expressed as a power series (Fan, 1978, 1979)

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$$h(B) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} B^k$$

Which converges in the norm topology. Let $S_m^*(r, \gamma)$ and $C_m(r, \gamma)$ denote the subclasses of $S(r, m)$ consisting of multivalent starlike and convex functions of order γ where $0 < \gamma < r$. A function $h \in S(r, m)$ belongs to these classes if it satisfies the conditions,

$$\Re \left\{ \frac{\zeta h'(\zeta)}{h(\zeta)} \right\} > \gamma \text{ and } \Re \left\{ 1 + \frac{\zeta h''(\zeta)}{h'(\zeta)} \right\} > \gamma$$

Respectively as introduced by Silverman (1975). Moreover, it is established (Goodman, 1983; Owa, 1992) that

$$h(\zeta) \in C_m(r, \gamma) \text{ if and only if } \frac{\zeta h'(\zeta)}{r} \in S_m^*(r, \gamma).$$

In Geometric Function Theory, elementary distributions such as Poisson, Binomial and Pascal have been investigated from a theoretical perspective (Altinkaya & Yalcin, 2017; El-Deeb, Bulboaca, & Dziok, 2019; Nazeer, Mehmood, Kang, & Har, 2019).

A discrete random variable x is said to have a Borel distribution if it takes the values $1, 2, 3, \dots$ with the probabilities $\frac{e^{-\rho}}{1!}, \frac{2\rho e^{-2\rho}}{2!}, \frac{9\rho^2 e^{-3\rho}}{3!}, \dots$ respectively, where ρ is the positive parameter.

Wanas and Khuttar (2020) defined the Borel distribution whose probability mass function is

$$P(x = u) = \frac{(\rho u)^{u-1} e^{-\rho u}}{u!}, \quad u = 1, 2, 3, \dots$$

Wanas *et al* (2020) introduced the power series whose coefficients correspond to Borel distribution probabilities (BD) as

$$\begin{aligned} N_r(\rho, \zeta) &= \zeta^r + \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!} \\ &= \zeta^r + \sum_{k=r+m}^{\infty} \varphi_{k,r}(\rho) \zeta^k \end{aligned} \quad (1.3)$$

where $0 < \rho \leq 1$ and

$$\varphi_{k,r}(\rho) = \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!}.$$

We consider a linear operator $M_r(\rho, \zeta) h(\zeta): T(r, m) \rightarrow T(r, m)$ defined by convolution (Hadamard product)

$$\begin{aligned} M_r(\rho, \zeta) h(\zeta) &= N_r(\rho, \zeta) * h(\zeta) \\ &= \zeta^r - \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!} b_k \zeta^k \end{aligned} \quad (1.4)$$

where $b_k \geq 0, 0 < \rho \leq 1$ and $\zeta \in \mathcal{U}$.

If $h(B)$ is defined by (1.2), then

$$\begin{aligned} M_r(\rho, B) h(B) &= N_r(\rho, B) * h(B) \\ &= B^r - \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!} b_k B^k \end{aligned} \quad (1.5)$$

Definition 1. (Owa, 1978) The fractional integral operator of order $\vartheta (\vartheta > 0)$ is defined by

$$D_B^{-\vartheta} = \frac{1}{\Gamma(\vartheta)} \int_0^1 \frac{B^{\vartheta} h(tB)}{(1-t)^{1-\vartheta}} dt \quad (1.6)$$

where h is analytic in a simply connected region of the complex plane containing the origin.

Definition 2. (Owa, 1978) The fractional derivative operator of order $\vartheta (0 \leq \vartheta < 1)$ is defined by

$$D_B^{\vartheta} = \frac{1}{\Gamma(1-\vartheta)} \frac{d}{dt} \int_0^1 \frac{B^{1-\vartheta} h(tB)}{(1-t)^{\vartheta}} dt \quad (1.7)$$

where h is analytic in a simply connected region of the complex plane containing the origin.

Applying Definitions 1.1 and 1.2 to functions $h \in T(r, m)$, we obtain

$$D_B^{-\vartheta} = \frac{\Gamma(r+1)}{\Gamma(r+\vartheta+1)} B^{r+\vartheta} - \frac{\Gamma(k+1)}{\Gamma(k+\vartheta+1)} b_k B^{k+\vartheta} \quad (1.8)$$

and

$$D_B^{\vartheta} = \frac{\Gamma(r+1)}{\Gamma(r-\vartheta+1)} B^{r-\vartheta} - \frac{\Gamma(k+1)}{\Gamma(k-\vartheta+1)} b_k B^{k-\vartheta} \quad (1.9)$$

The operator on Hilbert Space was recently considered by few researchers (Chrakim, Lee, & Lee, 1998; Ghanim & Darus, 2008; Rosy, 2001). This article focuses on analyzing the geometric properties of two specific subclasses, namely $TS_r^*(\gamma, \delta, \eta, B)$ and $TK_r(\gamma, \delta, \eta, B)$.

Main Results

The Subclass $TS_r^*(\gamma, \delta, \eta, B)$.

Definition 1. A function $h \in T(r, m)$ is said to be in the class $TS_r^*(\gamma, \delta, \eta, B)$ if it satisfies the inequality.

$$\|B(M_r(\rho, B)h(B))' - r(M_r(\rho, B)h(B))\| < \delta \| \gamma B(M_r(\rho, B)h(B))' - (r - \eta)(M_r(\rho, B)h(B)) \| \quad (2.1)$$

for $0 \leq \gamma \leq 1, 0 < \delta \leq 1, 0 \leq \eta < r, r \in \mathbb{N}$ and all operators B with $\|B\| < 1$ and $B \neq \emptyset$, where $\emptyset$ is the zero operator on $\mathcal{H}$

Theorem 1. Let $h \in T(r, m)$, then $h \in TS_r^*(\gamma, \delta, \eta, B)$ for all $B \neq \emptyset$ if and only if

$$\sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(k\gamma + r - \eta)]}{(k-r)!} b_k \leq \delta(\gamma r + r - \eta)$$

where $0 \leq \gamma \leq 1, 0 < \delta \leq 1, 0 \leq \eta < r, r \in \mathbb{N}, k \geq 2$ and $0 < \rho \leq 1$. Sharpness for the result is given by the function

$$h(\zeta) = \zeta^r - \frac{\delta(\gamma r + r - \eta)(k-r)!}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(k\gamma + r - \eta)]} \zeta^k, \quad k \geq 2 \quad (2.2)$$

Proof. Suppose $h \in TS_r^*(\gamma, \delta, \eta, B)$, then

$$\begin{aligned} & \|B(M_r(\rho, B)h(B))' - r(M_r(\rho, B)h(B))\| \\ & - \delta \| \gamma B(M_r(\rho, B)h(B))' - (r - \eta)(M_r(\rho, B)h(B)) \| \\ & = \left\| - \sum_{k=r+m}^{\infty} \frac{(k-r)(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!} b_k B^k \right\| \\ & - \delta \| (\gamma r + r - \eta) B^r \\ & - \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} (k\gamma + r - \eta)}{(k-r)!} b_k B^k \| \\ & \leq \sum_{k=r+m}^{\infty} \frac{(k-r)(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!} b_k \|B\|^k \\ & - \delta(\gamma r + r - \eta) \|B\|^r \\ & + \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} \delta(k\gamma + r - \eta)}{(k-r)!} b_k \|B\|^k \\ & \leq \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(k\gamma + r - \eta)]}{(k-r)!} b_k \\ & - \delta(\gamma r + r - \eta) \leq 0 \\ & \Rightarrow h \in TS_r^*(\gamma, \delta, \eta, B). \end{aligned}$$

To establish the converse, assume $h \in TS_r^*(\gamma, \delta, \eta, B)$, then

$$\|B(M_r(\rho, B)h(B))' - r(M_r(\rho, B)h(B))\| < \delta \| \gamma B(M_r(\rho, B)h(B))' - (r - \eta)(M_r(\rho, B)h(B)) \|$$

$$\begin{aligned} & \Rightarrow \left\| - \sum_{k=r+m}^{\infty} \frac{(k-r)(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!} b_k B^k \right\| \\ & < \delta \| (\gamma r + r - \eta) B^r \\ & - \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} (k\gamma + r - \eta)}{(k-r)!} b_k B^k \| \end{aligned}$$

Set $B = sI$ ($0 < s < 1$),

$$\frac{\sum_{k=r+m}^{\infty} \frac{(k-r)(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!} b_k s^k}{(\gamma r + r - \eta) s^r - \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} (k\gamma + r - \eta)}{(k-r)!} b_k s^k} < \delta. \quad (2.3)$$

By clearing the denominator and letting $s \rightarrow 1$, we obtain

$$\begin{aligned} & \sum_{k=r+m}^{\infty} \frac{(k-r)(\rho(k-r))^{k-r-1} e^{-\rho(k-r)}}{(k-r)!} b_k \\ & < \delta(\gamma r + r - \eta) \\ & - \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} \delta(k\gamma + r - \eta)}{(k-r)!} b_k. \end{aligned}$$

Thus,

$$\sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(k\gamma + r - \eta)]}{(k-r)!} b_k \leq \delta(\gamma r + r - \eta).$$

Corollary 1. If $h \in TS_r^*(\gamma, \delta, \eta, B)$, then

$$b_k \leq \frac{\delta(\gamma r + r - \eta)(k-r)!}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(k\gamma + r - \eta)]}, \quad k \geq 2. \quad (2.4)$$

Theorem 2. If $h \in TS_r^*(\gamma, \delta, \eta, B)$, then

$$\begin{aligned} & \|D_B^{-\vartheta} h(B)\| \\ & \leq \frac{\Gamma(r+1)}{\Gamma(r+\vartheta+1)} \|B\|^{r+\vartheta} \left[1 \right. \\ & \left. + \frac{\delta(\gamma r + r - \eta) m! \Gamma(r+m+1) \Gamma(r+\vartheta+1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m+\vartheta+1)} \right] \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} & \|D_B^{-\vartheta} h(B)\| \\ & \geq \frac{\Gamma(r+1)}{\Gamma(r+\vartheta+1)} \|B\|^{r+\vartheta} \left[1 \right. \\ & \left. - \frac{\delta(\gamma r + r - \eta) m! \Gamma(r+m+1) \Gamma(r+\vartheta+1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m+\vartheta+1)} \right]. \end{aligned} \quad (2.6)$$

Sharpness for the result is given by the function

$$h(\zeta) = \zeta^r - \frac{\delta(\gamma r + r - \eta)m!}{(\rho m)^{m-1}e^{-\rho m}[m + \delta(k\gamma + r - \eta)]} \zeta^{r+m} \quad (r, m \in \mathbb{N}). \quad (2.7)$$

Proof: Suppose that $h \in TS_r^*(\gamma, \delta, \eta, B)$ then by (1.8) we have

$$\begin{aligned} & \frac{\Gamma(r + \vartheta + 1)}{\Gamma(r + 1)} B^{-\vartheta} D_B^{-\vartheta} h(B) \\ &= B^r - \sum_{k=r+m}^{\infty} \frac{\Gamma(k+1)\Gamma(r + \vartheta + 1)}{\Gamma(r+1)\Gamma(k + \vartheta + 1)} b_k B^k \\ &= B^r - \sum_{k=r+m}^{\infty} \psi(k, \vartheta) b_k B^k, \end{aligned}$$

Where

$$\psi(k, \vartheta) = \frac{\Gamma(k+1)\Gamma(r + \vartheta + 1)}{\Gamma(r+1)\Gamma(k + \vartheta + 1)}, \quad (k \geq r + m, r, m \in \mathbb{N}).$$

Since for $k \geq r + m$, as the value of k increases $\psi(k, \vartheta)$ decreases

$$0 < \psi(k, \vartheta) \leq \psi(r + m, \vartheta) = \frac{\Gamma(k+1)\Gamma(r + \vartheta + 1)}{\Gamma(r+1)\Gamma(k + \vartheta + 1)} \quad (2.8)$$

Using Theorem 2.1 and (2.8), we get

$$\begin{aligned} & \left\| \frac{\Gamma(r + \vartheta + 1)}{\Gamma(r + 1)} B^{-\vartheta} D_B^{-\vartheta} h(B) \right\| \\ & \leq \|B\|^r + \sum_{k=r+m}^{\infty} \psi(k, \vartheta) b_k \|B\|^k \\ & \leq \|B\|^r + \psi(r + m, \vartheta) \|B\|^{r+m} \sum_{k=r+m}^{\infty} b_k \\ & \leq \|B\|^r \\ & + \frac{\delta(\gamma r + r - \eta)m! \Gamma(r + m + 1) \Gamma(r + \vartheta + 1) \|B\|^{r+m}}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r + 1) \Gamma(r + m + \vartheta + 1)} \end{aligned}$$

Multiplying both the sides by $\|B\|^\vartheta$ and dividing by the scalar $\Gamma(r + \vartheta + 1)/\Gamma(r + 1)$, we obtain

$$\begin{aligned} & \|D_B^{-\vartheta} h(B)\| \\ & \leq \frac{\Gamma(r + 1)}{\Gamma(r + \vartheta + 1)} \|B\|^{r+\vartheta} \left[1 \right. \\ & \quad \left. + \frac{\delta(\gamma r + r - \eta)m! \Gamma(r + m + 1) \Gamma(r + \vartheta + 1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r + 1) \Gamma(r + m + \vartheta + 1)} \right]. \end{aligned}$$

Also

$$\begin{aligned} & \left\| \frac{\Gamma(r + \vartheta + 1)}{\Gamma(r + 1)} B^{-\vartheta} D_B^{-\vartheta} h(B) \right\| \\ & \geq \|B\|^r - \sum_{k=r+m}^{\infty} \psi(k, \vartheta) b_k \|B\|^k \\ & \geq \|B\|^r - \psi(r + m, \vartheta) \|B\|^{r+m} \sum_{k=r+m}^{\infty} b_k \\ & \geq \|B\|^r - \frac{\delta(\gamma r + r - \eta)m! \Gamma(r + m + 1) \Gamma(r + \vartheta + 1) \|B\|^{r+m}}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r + 1) \Gamma(r + m + \vartheta + 1)} \end{aligned}$$

Multiplying both the sides by $\|B\|^\vartheta$ and dividing by the scalar $\Gamma(r + \vartheta + 1)/\Gamma(r + 1)$, we obtain

$$\begin{aligned} & \|D_B^{-\vartheta} h(B)\| \\ & \geq \frac{\Gamma(r + 1)}{\Gamma(r + \vartheta + 1)} \|B\|^{r+\vartheta} \left[1 \right. \\ & \quad \left. - \frac{\delta(\gamma r + r - \eta)m! \Gamma(r + m + 1) \Gamma(r + \vartheta + 1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r + 1) \Gamma(r + m + \vartheta + 1)} \right]. \end{aligned}$$

A direct consequence of Theorem 2.2 by taking $\vartheta = 1$ is given below.

Corollary 2. If $h \in TS_r^*(\gamma, \delta, \eta, B)$, then

$$\begin{aligned} & \left\| \int_0^1 B h(tB) dt \right\| \\ & \leq \frac{1}{(r+1)} \|B\|^{r+1} \left[1 \right. \\ & \quad \left. + \frac{\delta(\gamma r + r - \eta)m! (r+1)}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] (r+m+1)} \|B\|^m \right] \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_0^1 B h(tB) dt \right\| \\ & \geq \frac{1}{(r+1)} \|B\|^{r+1} \left[1 \right. \\ & \quad \left. - \frac{\delta(\gamma r + r - \eta)m! (r+1)}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] (r+m+1)} \|B\|^m \right] \end{aligned}$$

Theorem 3. If $h \in TS_r^*(\gamma, \delta, \eta, B)$, then

$$\begin{aligned} & \|D_B^\vartheta h(B)\| \\ & \leq \frac{\Gamma(r + 1)}{\Gamma(r - \vartheta + 1)} \|B\|^{r-\vartheta} \left[1 \right. \\ & \quad \left. + \frac{\delta(\gamma r + r - \eta)m! \Gamma(r + m + 1) \Gamma(r - \vartheta + 1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r + 1) \Gamma(r + m - \vartheta + 1)} \right] \quad (2.11) \end{aligned}$$

and

$$\begin{aligned} & \|D_B^\vartheta h(B)\| \\ & \geq \frac{\Gamma(r + 1)}{\Gamma(r - \vartheta + 1)} \|B\|^{r-\vartheta} \left[1 \right. \\ & \quad \left. - \frac{\delta(\gamma r + r - \eta)m! \Gamma(r + m + 1) \Gamma(r - \vartheta + 1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r + 1) \Gamma(r + m - \vartheta + 1)} \right] \quad (2.12) \end{aligned}$$

Sharpness for the result is given by (2.7)

Proof: For $h \in TS_r^*(\gamma, \delta, \eta, B)$, it follows from (1.9) that

$$\begin{aligned} & \frac{\Gamma(r - \vartheta + 1)}{\Gamma(r + 1)} B^\vartheta D_B^\vartheta h(B) = B^r - \sum_{k=r+m}^{\infty} \frac{\Gamma(k+1)\Gamma(r - \vartheta + 1)}{\Gamma(r+1)\Gamma(k - \vartheta + 1)} b_k B^k \\ & = B^r - \sum_{k=r+m}^{\infty} \phi(k, \vartheta) b_k B^k, \end{aligned}$$

Where

$$\phi(k, \vartheta) = \frac{\Gamma(k+1)\Gamma(r-\vartheta+1)}{\Gamma(r+1)\Gamma(k-\vartheta+1)}.$$

Since for $k \geq r+m$, as the value of k increases $\phi(k, \vartheta)$ decreases

$$0 < \phi(k, \vartheta) \leq \phi(r+m, \vartheta) = \frac{\Gamma(k+1)\Gamma(r-\vartheta+1)}{\Gamma(r+1)\Gamma(k-\vartheta+1)} \quad (2.13)$$

An application of Theorem 2.1 and (2.13) gives

$$\begin{aligned} & \left\| \frac{\Gamma(r-\vartheta+1)}{\Gamma(r+1)} B^\vartheta D_B^\vartheta h(B) \right\| \\ & \leq \|B\|^r + \sum_{k=r+m}^{\infty} \phi(k, \vartheta) b_k \|B\|^k \\ & \leq \|B\|^r + \phi(r+m, \vartheta) \|B\|^{r+m} \sum_{k=r+m}^{\infty} b_k \\ & \leq \|B\|^r + \frac{\delta(\gamma r + r - \eta) m! \Gamma(r+m+1) \Gamma(r-\vartheta+1) \|B\|^{r+m}}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m-\vartheta+1)}. \end{aligned}$$

Multiplying both the sides by $\|B\|^{-\vartheta}$ and dividing by the scalar $\Gamma(r-\vartheta+1)/\Gamma(r+1)$, we obtain

$$\begin{aligned} & \|D_B^\vartheta h(B)\| \\ & \leq \frac{\Gamma(r+1)}{\Gamma(r-\vartheta+1)} \|B\|^{r-\vartheta} \left[1 \right. \\ & \quad \left. + \frac{\delta(\gamma r + r - \eta) m! \Gamma(r+m+1) \Gamma(r-\vartheta+1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m-\vartheta+1)} \right]. \end{aligned}$$

Also

$$\begin{aligned} & \left\| \frac{\Gamma(r-\vartheta+1)}{\Gamma(r+1)} B^\vartheta D_B^\vartheta h(B) \right\| \\ & \geq \|B\|^r - \sum_{k=r+m}^{\infty} \phi(k, \vartheta) b_k \|B\|^k \\ & \geq \|B\|^r - \phi(r+m, \vartheta) \|B\|^{r+m} \sum_{k=r+m}^{\infty} b_k \\ & \geq \|B\|^r - \frac{\delta(\gamma r + r - \eta) m! \Gamma(r+m+1) \Gamma(r-\vartheta+1) \|B\|^{r+m}}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m-\vartheta+1)}. \end{aligned}$$

Multiplying both the sides by $\|B\|^{-\vartheta}$ and dividing by the scalar $\Gamma(r-\vartheta+1)/\Gamma(r+1)$, we obtain

$$\begin{aligned} & \|D_B^\vartheta h(B)\| \\ & \geq \frac{\Gamma(r+1)}{\Gamma(r-\vartheta+1)} \|B\|^{r-\vartheta} \left[1 \right. \\ & \quad \left. - \frac{\delta(\gamma r + r - \eta) m! \Gamma(r+m+1) \Gamma(r-\vartheta+1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [m + \delta(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m-\vartheta+1)} \right]. \end{aligned}$$

Theorem 4. Let $h_r(\zeta) = \zeta^r$ and

$$h_k(\zeta) = \zeta^r - \frac{\delta(\gamma r + r - \eta)(k-r)!}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(kr + r - \eta)]} \zeta^k$$

then $h \in TS_r^*(\gamma, \delta, \eta, B)$ if and only if it can be expressed in the form:

$$h(\zeta) = \Theta_r \zeta^r + \sum_{k=r+m}^{\infty} \Theta_k h_k(\zeta) \quad (2.14)$$

where $\Theta_r \geq 0, \Theta_k \geq 0, k \geq r+m$ and $\Theta_r + \sum_{k=r+m}^{\infty} \Theta_k = 1$

Proof: If h is expressed in the form (2.14), then

$$\begin{aligned} & h(\zeta) \\ & = \Theta_r \zeta^r \\ & + \sum_{k=r+m}^{\infty} \Theta_k \left[\zeta^r \right. \\ & \quad \left. - \frac{\delta(\gamma r + r - \eta)(k-r)! \zeta^k}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(kr + r - \eta)]} \right] \\ & = \zeta^r - \sum_{k=r+m}^{\infty} \Theta_k \frac{\delta(\gamma r + r - \eta)(k-r)!}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(kr + r - \eta)]} \zeta^k \end{aligned}$$

Hence,

$$\begin{aligned} & \sum_{k=r+m}^{\infty} \frac{\delta(\gamma r + r - \eta)(k-r)!}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(kr + r - \eta)]} \\ & \times \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(kr + r - \eta)]}{\delta(\gamma r + r - \eta)(k-r)!} \Theta_k \\ & = \sum_{k=r+m}^{\infty} \Theta_k = 1 - \Theta_r \leq 1 \\ & \Rightarrow h \in TS_r^*(\gamma, \delta, \eta, B). \end{aligned}$$

Conversely, suppose $h \in TS_r^*(\gamma, \delta, \eta, B)$, then

$$\Theta_k = \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(kr + r - \eta)]}{\delta(\gamma r + r - \eta)(k-r)!} b_k$$

where b_k is given by (2.4). Then

$$\begin{aligned} & h(\zeta) = \zeta^r + \\ & \quad \sum_{k=r+m}^{\infty} \frac{\delta(\gamma r + r - \eta)(k-r)! \Theta_k \zeta^k}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(k\gamma + r - \eta)]} \\ & = \zeta^r + \sum_{k=r+m}^{\infty} (\zeta^r - h_k(\zeta)) \Theta_k \\ & = \left(1 - \sum_{k=r+m}^{\infty} \Theta_k \right) \zeta^r + \sum_{k=r+m}^{\infty} \Theta_k h_k(\zeta) \\ & = \Theta_r \zeta^r + \sum_{k=r+m}^{\infty} \Theta_k h_k(\zeta). \end{aligned}$$

Theorem 5. The class $TS_r^*(\gamma, \delta, \eta, B)$ is closed under convex combination.

Proof. For $n = 1, 2, \dots$ and $h_n \in$

$$TS_r^*(\gamma, \delta, \eta, B), \quad h_n(\zeta) = \zeta^r - \sum_{k=r+m}^{\infty} b_{k,n} \zeta^k,$$

we have

$$\begin{aligned} & (\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [(k-r) + \delta(kr + r - \eta)] \\ & \leq \delta(kr + r - \eta)(k-r)!. \end{aligned} \quad (2.15)$$

For $\sum_{n=1}^{\infty} \eta_n = 1, 0 \leq \eta_n \leq 1$, the convex combination of h_n , may be written as

$$\sum_{n=1}^{\infty} \eta_n h_n(\zeta) = \zeta^r - \sum_{k=r+m}^{\infty} \left(\sum_{n=1}^{\infty} \eta_n b_{k,n} \right) \zeta^k.$$

By (2.15), we get

$$\begin{aligned} & \sum_{k=r+m}^{\infty} (\rho(k-r)^{k-r-1} e^{-\rho(k-r)} [(k-r) \\ & \quad + \delta(k\gamma + r - \eta)]) \left(\sum_{n=1}^{\infty} \eta_n b_{k,n} \right) \\ &= \sum_{n=1}^{\infty} \eta_n \left(\sum_{k=r+m}^{\infty} (\rho(k-r)^{k-r-1} e^{-\rho(k-r)} [(k-r) \right. \\ & \quad \left. + \delta(k\gamma + r - \eta)]) b_{k,n} \right) \\ &\leq \sum_{n=1}^{\infty} \eta_n (\delta(\gamma r + r - \eta)(k-r)! = \delta(\gamma r + r - \eta)(k-r)!) \\ &\quad \Rightarrow \sum_{n=1}^{\infty} \eta_n h_n(\zeta) \in TS_r^*(\gamma, \delta, \eta, B). \end{aligned}$$

Corollary 3. The class $TS_r^*(\gamma, \delta, \eta, B)$ is a convex set.

The Subclass $TK_r(\gamma, \delta, \eta, B)$.

Definition 2. A function $h \in T(r, m)$ is said to be in the class $TK_r(\gamma, \delta, \eta, B)$ if it satisfies the inequality

$$\begin{aligned} & \|B(M_r(\rho, B)h(B))'' + (1-r)(M_r(\rho, B)h(B))'\| < \\ & \delta \| \gamma B(M_r(\rho, B)h(B))'' - (r-\eta + \\ & \gamma)(M_r(\rho, B)h(B))' \| \end{aligned} \quad (2.16)$$

for $0 \leq \gamma \leq 1, 0 < \delta \leq 1, 0 \leq \eta < r, r \in \mathbb{N}$ and all operators B with $\|B\| < 1$ and $B \neq \emptyset$, where $\emptyset$ is the zero operator on $\mathcal{H}$.

Theorem 6. Let $h \in T(r, m)$ be defined by (1.2). Then $h \in TK_r(\gamma, \delta, \eta, B)$ for all $B \neq \emptyset$ if and only if

$$\begin{aligned} & \sum_{k=r+m}^{\infty} \frac{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [k(k-r) + \delta k(k\gamma + r - \eta)] b_k}{r(k-r)!} \\ & \leq \delta(\gamma r + r - \eta) \end{aligned}$$

where $0 \leq \gamma \leq 1, 0 < \delta \leq 1, 0 \leq \eta < r, r \in \mathbb{N}, k \geq 2$ and $0 < \rho \leq 1$. Sharpness for the result is given by

$$\begin{aligned} & h(\zeta) = \zeta^r \\ & - \frac{\delta r(\gamma r + r - \eta)(k-r)!}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [k(k-r) + \delta k(k\gamma + r - \eta)]} \zeta^k. \end{aligned} \quad (2.17)$$

Corollary 4. If $h \in TK_r(\gamma, \delta, \eta, B)$, then

$$b_k \leq \frac{\delta r(\gamma r + r - \eta)(k-r)!}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [k(k-r) + \delta k(k\gamma + r - \eta)]}, \quad k \geq 2. \quad (2.18)$$

Theorem 7. If $h \in TK_r(\gamma, \delta, \eta, B)$, then $\|D_B^{-\vartheta} h(B)\|$

$$\begin{aligned} & \leq \frac{\Gamma(r+1)}{\Gamma(r+\vartheta+1)} \|B\|^{r+\vartheta} \left[1 \right. \\ & \quad \left. + \frac{\delta r(\gamma r + r - \eta)m! \Gamma(r+m+1) \Gamma(r+\vartheta+1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [mk + \delta k(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m+\vartheta+1)} \right] \end{aligned} \quad (2.19)$$

and

$$\begin{aligned} & \|D_B^{-\vartheta} h(B)\| \\ & \geq \frac{\Gamma(r+1)}{\Gamma(r+\vartheta+1)} \|B\|^{r+\vartheta} \left[1 \right. \\ & \quad \left. - \frac{\delta r(\gamma r + r - \eta)m! \Gamma(r+m+1) \Gamma(r+\vartheta+1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [mk + \delta k(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m+\vartheta+1)} \right] \end{aligned} \quad (2.20)$$

Sharpness for the result is given by

$$\begin{aligned} & h(\zeta) \\ & = \zeta^r - \frac{\delta r(\gamma r + r - \eta)m!}{(\rho m)^{m-1} e^{-\rho m} [mk + \delta k(k\gamma + r - \eta)]} \zeta^{r+m}, \end{aligned} \quad (r, m \in \mathbb{N}). \quad (2.21)$$

A direct consequence of the Theorem 2.7 by taking $\vartheta = 1$ is given below.

Corollary 5. If $h \in TK_r(\gamma, \delta, \eta, B)$, then

$$\begin{aligned} & \left\| \int_0^1 B h(tB) dt \right\| \\ & \leq \frac{1}{(r+1)} \|B\|^{r+1} \left[1 \right. \\ & \quad \left. + \frac{\delta r(\gamma r + r - \eta)m! (r+1)}{(\rho m)^{m-1} e^{-\rho m} [mk + \delta k(k\gamma + r - \eta)] (r+m+1)} \|B\|^m \right] \end{aligned}$$

and

$$\begin{aligned} & \left\| \int_0^1 B h(tB) dt \right\| \\ & \geq \frac{1}{(r+1)} \|B\|^{r+1} \left[1 \right. \\ & \quad \left. - \frac{\delta(\gamma r + r - \eta)m! (r+1)}{(\rho m)^{m-1} e^{-\rho m} [mk + \delta k(k\gamma + r - \eta)] (r+m+1)} \|B\|^m \right]. \end{aligned}$$

Theorem 8. If $h \in TK_r(\gamma, \delta, \eta, B)$, then

$$\begin{aligned} & \|D_B^{\vartheta} h(B)\| \\ & \leq \frac{\Gamma(r+1)}{\Gamma(r-\vartheta+1)} \|B\|^{r-\vartheta} \left[1 \right. \\ & \quad \left. + \frac{\delta r(\gamma r + r - \eta)m! \Gamma(r+m+1) \Gamma(r-\vartheta+1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [mk + \delta k(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m-\vartheta+1)} \right] \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} & \|D_B^\vartheta h(B)\| \\ & \geq \frac{\Gamma(r+1)}{\Gamma(r-\vartheta+1)} \|B\|^{r-\vartheta} \left[1 - \frac{\delta r(\gamma r + r - \eta) m! \Gamma(r+m+1) \Gamma(r-\vartheta+1) \|B\|^m}{(\rho m)^{m-1} e^{-\rho m} [mk + \delta k(k\gamma + r - \eta)] \Gamma(r+1) \Gamma(r+m-\vartheta+1)} \right] \end{aligned} \quad (2.23)$$

Sharpness for the result is given by (2.21)

Theorem 9. Let $h_r(\zeta) = \zeta^r$ and

$$h_k(\zeta) = \zeta^r - \frac{\delta r(\gamma r + r - \eta)(k-r)!}{(\rho(k-r))^{k-r-1} e^{-\rho(k-r)} [k(k-r) + \delta k(k\gamma + r - \eta)]} \zeta^k,$$

where $k \geq r + m$. Then $h \in TK_r(\gamma, \delta, \eta, B)$ if and only if it can be expressed in the form:

$$h(\zeta) = \theta_r \zeta^r + \sum_{k=r+m}^{\infty} \theta_k h_k(\zeta) \quad (2.24)$$

where $(\theta_r \geq 0, \theta_k \geq 0, k \geq r + m)$ and

$$\theta_r + \sum_{k=r+m}^{\infty} \theta_k = 1.$$

Theorem 10. The class $TK_r(\gamma, \delta, \eta, B)$ is closed under convex combination.

Corollary 6. The class $TK_r(\gamma, \delta, \eta, B)$ is a convex set.

Results of the above theorems and corollaries in $TK_r(\gamma, \delta, \eta, B)$ can be proved akin to the subclass $TS_r^*(\gamma, \delta, \eta, B)$.

Conclusions

This research article deals with coefficient estimates, extreme points and convex combination of functions in the newly defined two subclasses $TS_r^*(\gamma, \delta, \eta, B)$ and $TK_r(\gamma, \delta, \eta, B)$. It also opens new avenues for the analytical treatment of multivalent function classes in complex Hilbert spaces using operator- theoretic and fractional calculus techniques. The underlying structure invites further generalisation and gives a potential future direction which might include investigating the structural implication of fractional differential subordination, also formulating analogues of classical distortion, growth and covering theorems under the influence of fractional and distributional operators.

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