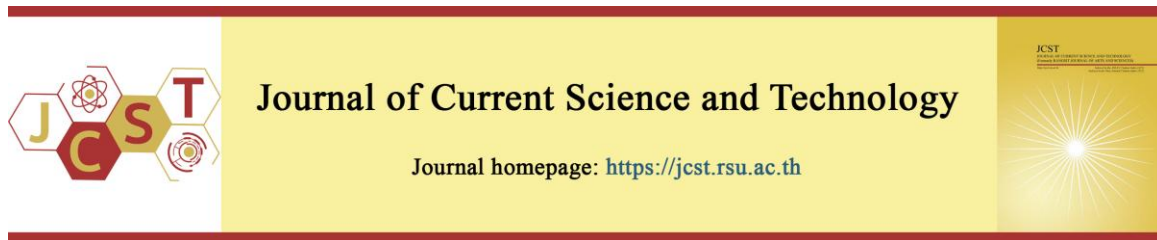


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Applying Bias-Correction Methods to Parameter Estimation for the Zeghdoudi Distribution in Medical Data

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Abstract

The Zeghdoudi distribution (ZD) is a valuable model for analyzing lifetime data. It has been further developed for modeling various data types and for constructing related lifetime models. In statistical inference, the maximum likelihood method is well known for estimating the parameters of different distributions using large samples. However, when applied to small or moderate samples, the bias of the maximum likelihood estimator (MLE) can become problematic. This study proposes point estimators for the Zeghdoudi distribution using two bias-correction methods: the Cox-Snell method and the parametric bootstrap. The performance of these estimators is evaluated using average bias and root mean square error (RMSE) in simulation studies. The results demonstrate that the point estimators derived from the bias-correction methods improve finite-sample accuracy by reducing the bias of the maximum likelihood estimator to second order. In addition, the findings indicate that the parametric bootstrap technique outperformed the others under various conditions, whereas the MLE can be biased for small or moderate sample sizes. Furthermore, analyses of two medical datasets were conducted to validate the performance of the proposed estimators.

Keywords: *bias-correction, lifetime distribution, maximum likelihood estimator, point estimation*

1. Introduction

Statistical models are essential for describing data behavior across diverse fields, including healthcare, economics, engineering, and related areas. Statistical analysis of lifetime data has become an important area of research across numerous disciplines. Lifetime distribution methodologies are used in a wide range of applications, such as product durability assessment and medical research involving disease progression and treatment outcomes. In the context of engineering and medical sciences, Messaadia and Zeghdoudi (2018) introduced a one-parameter lifetime distribution known as the Zeghdoudi distribution (ZD). The ZD is derived from a mixture of two classical gamma distributions,

specifically gamma $(2, \theta)$ and gamma $(3, \theta)$, and has been shown to outperform several competing models.

The probability density function (PDF) and cumulative distribution function (CDF) of the ZD are given as follows:

$$f(x; \theta) = \frac{\theta^3 x(1+x)e^{-\theta x}}{2+\theta}; x, \theta > 0, \quad (1)$$

and,

$$F(x; \theta) = 1 - \left(\frac{x^2 \theta^2 + \theta(\theta+2)x + \theta+2}{\theta+2} \right) e^{-\theta x}; x, \theta > 0.$$

The shapes of the PDFs of the ZD for different parameter values are shown in Figure 1.

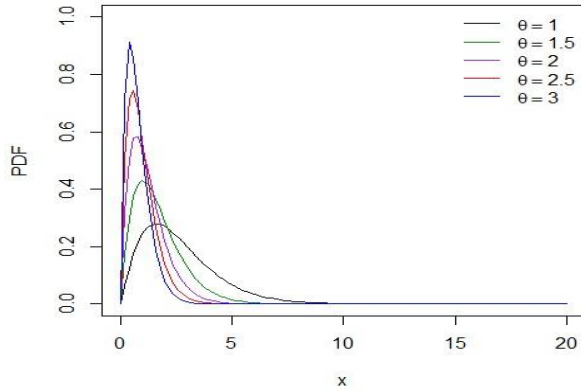


Figure 1 The PDFs of the ZD for different parameter values

Existing research on ZD has followed two main directions. The first generalizes the distribution to increase its flexibility, including the transmuted ZD (Ruidas, 2020), the power ZD (Aidi et al., 2022), the inverse power ZD (Elbatal et al., 2024), and the unit ZD (Bashiru et al., 2025). The second concerns interval estimation, in which Tulyanitikul and Panichkitkosolkul (2025) constructed confidence intervals for the ZD parameter. Both directions, however, rely on the maximum likelihood estimator of θ . Point estimation is one of the most widely used forms of statistical inference, and maximum likelihood is its most common approach; while its estimators have desirable large-sample properties, for small and moderate samples, the regime that characterizes most medical lifetime data, the maximum likelihood estimator (MLE) can be substantially biased. To date, and to the best of our knowledge, no study has attempted to reduce the bias of the maximum likelihood estimator of the ZD parameter. Because the single parameter θ governs the entire density, shape, and hazard behavior of the ZD, a biased estimate of θ propagates directly into biased reliability and survival estimates, which can mislead conclusions drawn from small clinical datasets. This motivates the development of bias-corrected point estimators for the ZD.

Analytical and resampling-based bias corrections have already proven effective for many other lifetime distributions, where they consistently reduced both the bias and the root mean squared error (RMSE) of the MLE relative to the uncorrected estimator. Such studies include the generalized Ramos-Louzada distribution, the inverse Xgamma distribution, the unit odd Lindley half-logistic distribution, the two-parameter Lindley distribution, the generalized normal distribution, the

Kumaraswamy distribution, the modified power function distribution, and the extended Rama distribution (Ahmed & Algamal, 2025; Ahmed et al., 2024; Gemeay & Abd El-Bar, 2025; Gül, 2025; Gül & Doğru, 2024; Lemonte, 2011; Sangpoom & Klomwises, 2021; Sungboonchoo, 2025). Despite this body of work, no comparable bias correction has been derived or evaluated for the Zeghdoudi distribution.

To achieve these objectives, two bias-correction methods are employed, both reducing the bias of the maximum likelihood estimator from first order, $O(n^{-1})$, to second order, $O(n^{-2})$. The first is a corrective estimator derived following Cox and Snell (1968), which removes the bias by subtracting its estimated value from the original MLE. The second is the parametric bootstrap of Efron (1982), which corrects the bias numerically and requires no explicit derivation of the bias function.

We conducted a Monte Carlo simulation to assess the performance of the three estimators of the ZD parameter, maximum likelihood, Cox-Snell, and parametric bootstrap, and then applied them to two real medical datasets to evaluate their practical performance. The remainder of this paper is organized as follows. Section 2 explains the objectives of this research. Section 3 presents the three estimation methods, describes the simulation design, and applies the estimators to two medical datasets. Section 4 reports the simulation and application results, and Section 5 concludes.

2. Objectives

The objective of this study is to develop and evaluate two bias-corrected point estimators based on the Cox-Snell method and the parametric bootstrap for the parameter of the Zeghdoudi distribution, in order to reduce the bias of the maximum likelihood estimator and improve the accuracy of parameter estimation in small and moderate samples, and to assess their performance through Monte Carlo simulation and applications to real medical datasets.

3. Methodology

3.1 Parameter Estimation

This section presents three methods for estimating the ZD parameter. We divide the discussion into two parts: the first part presents the classical method, namely maximum likelihood estimation; the second part describes two bias-correction methods used to minimize the bias present in the maximum likelihood estimator. These methods are the Cox-Snell method and the parametric bootstrap method.

3.1.1 Maximum Likelihood Estimation

To find the maximum likelihood estimator of the unknown parameter of the ZD, let $X_1, X_2, \dots, X_n$ be a random sample from the ZD with the PDF given in Equation (1). The log-likelihood function of θ is given by:

$$\log L(\theta) = 3n \log(\theta) - n \log(2+\theta) + \sum_{i=1}^n \log(x_i + x_i^2) - \theta \sum_{i=1}^n x_i \quad (2)$$

The derivative of the log-likelihood function in Equation (2) with respect to θ , in order to obtain the maximum likelihood estimator ($\hat{\theta}$) of θ , can be written as:

$$\frac{\partial}{\partial \theta} \log L(\theta) = \frac{3n}{\theta} - \frac{n}{2+\theta} - \sum_{i=1}^n x_i. \quad (3)$$

Since Equation (3) is nonlinear, there is no closed-form solution for the maximum likelihood estimator of θ . To obtain the maximum likelihood estimate, we use a numerical method and verify that the solution is maximum. Specifically, we use the maxLik package (Henningsen & Toomet, 2011) in the statistical software R to estimate the parameter via the Newton-Raphson algorithm.

3.1.2 Bias-Corrected Maximum Likelihood Estimators

We adjusted the maximum likelihood estimator to improve the accuracy of parameter estimation for the ZD by applying two commonly used bias-correction approaches. These two methods reduce the bias of the maximum likelihood estimator, thereby improving bias reduction from first order to second order.

3.1.2.1 Cox-Snell Method

In the literature, the Cox-Snell method is commonly referred to as the corrective approach introduced by Cox and Snell (1968). Suppose $\log L(\theta)$ is the log-likelihood function based on a sample of n observations and p -dimensional vector of unknown parameters $\theta = (\theta_1, \theta_2, \dots, \theta_p)$. The joint cumulants of the derivatives of $\log L(\theta)$ are defined as follows, where $i, j, l = 1, 2, \dots, p$:

$$m_{ij} = E \left[\frac{\partial^2 \log L(\theta)}{\partial \theta_i \partial \theta_j} \right], \quad (4)$$

$$m_{ijl} = E \left[\frac{\partial^3 \log L(\theta)}{\partial \theta_i \partial \theta_j \partial \theta_l} \right], \quad (5)$$

$$m_{ij,l} = E \left[\left(\frac{\partial^2 \log L(\theta)}{\partial \theta_i \partial \theta_j} \right) \left(\frac{\partial \log L(\theta)}{\partial \theta_l} \right) \right]. \quad (6)$$

The derivatives of cumulants are determined as follows:

$$m_{ij}^{(l)} = \frac{\partial m_{ij}}{\partial \theta_l}, \quad i, j, l = 1, 2, \dots, p. \quad (7)$$

Each expression in Equations (4)–(7) is assumed to be of order $O(n)$. Cox and Snell (1968) defined the bias of the s -th element of the maximum likelihood estimator of the parameter vector θ , under the condition that each sample in the data set is independent of the others but not necessarily identically distributed, as follows:

$$\text{Bias}(\hat{\theta}_s) = \sum_{i=1}^p \sum_{j=1}^p \sum_{l=1}^p m^{si} m^{jl} [0.5 m_{ijl} + m_{ij,l}] + O(n^{-2}), \quad (8)$$

where $s, i, j = 1, 2, \dots, p$, the term m^{ij} represents the (i, j) -th element of the inverse of the Fisher information matrix, and $M = [-m_{ij}]$ is the Fisher information matrix for θ .

Cordeiro and Klein (1994) demonstrated that the formula remains valid for non-independent sample data, even when the samples are not identically distributed. The expression in Equation (8) can be restated as:

$$\text{Bias}(\hat{\theta}_s) = \sum_{i=1}^p m^{si} \sum_{j=1}^p \sum_{l=1}^p [m_{ij}^{(l)} - 0.5 m_{ijl}] m^{jl} + O(n^{-2}), \quad (9)$$

where $s = 1, 2, \dots, p$. Define a matrix $A = [A^{(1)} | A^{(2)} | \dots | A^{(p)}]$ with elements specified by $A^{(l)} = [a_{ij}^{(l)}]$, where $a_{ij}^{(l)} = m_{ij}^{(l)} - 0.5 m_{ijl}$ for $i, j, l = 1, 2, \dots, p$. Therefore, the bias expression of $\hat{\theta}$ from Equation (9) can be written as:

$$\text{Bias}(\hat{\theta}) = M^{-1} A \cdot \text{vec}(M^{-1}) + O(n^{-2}). \quad (10)$$

Thus, the bias-corrected maximum likelihood estimator of θ can be obtained as:

$$\hat{\theta}_{CS} = \hat{\theta} - \hat{M}^{-1} \hat{A} \cdot \text{vec}(\hat{M}^{-1}), \quad (11)$$

where $\hat{\theta}$ denotes the maximum likelihood estimator of θ , $\hat{M} = M|_{\theta=\hat{\theta}}$, $\hat{A} = A|_{\theta=\hat{\theta}}$, and the vec operator produces a column vector from a matrix by sequentially stacking its columns. Accordingly, the bias of $\hat{\theta}_{CS}$ is of the second order ($O(n^{-2})$).

To proceed with the Cox-Snell method, we examine the parameter θ of the ZD with $p = 1$ to derive the bias-corrected maximum likelihood estimator. The following values are obtained:

$$m_{11} = \frac{-3n}{\theta^2} + \frac{n}{(2+\theta)^2}$$

$$m_{111} = \frac{6n}{\theta^3} - \frac{2n}{(2+\theta)^3}.$$

the Fisher information matrix for θ is given by:

$$M = \begin{bmatrix} \frac{3n}{\theta^2} - \frac{n}{(2+\theta)^2} \end{bmatrix}.$$

in addition, we have

$$m_{11}^{(1)} = m_{111} = \frac{6n}{\theta^3} - \frac{2n}{(2+\theta)^3}.$$

the element of $A^{(1)}$ is $a_{11}^{(1)} = 0.5 \left[\frac{6n}{\theta^3} - \frac{2n}{(2+\theta)^3} \right]$.

Then, $A = [A^{(1)}] = [a_{11}^{(1)}]$,

$$A = 0.5 \left[\frac{6n}{\theta^3} - \frac{2n}{(2+\theta)^3} \right].$$

Therefore, the adjusted maximum likelihood estimator given in Equation (11), represented by $\hat{\theta}_{CS}$, exhibits a second-order bias.

3.1.2.2 Parametric Bootstrap Method

A method for reducing bias in the maximum likelihood estimator is the parametric bootstrap resampling method (Efron, 1982). This technique employs maximum likelihood estimators derived from the data. It uses these estimators to generate random samples from the ZD, estimates the bias, and subsequently subtracts it from the maximum likelihood estimator.

The bias of the maximum likelihood estimator ($\hat{\theta}_{MLE}$) given in Equation (12), is:

$$\text{Bias}(\hat{\theta}_{MLE}) = \frac{1}{B} \sum_{r=1}^B \hat{\theta}_r - \hat{\theta}_{MLE}, \quad (12)$$

where $\hat{\theta}_r$ represents the MLE of θ from the r -th bootstrap sample generated from the ZD. Then, the parametric bootstrap bias-corrected estimator in Equation (13) is defined as:

$$\hat{\theta}_{PB} = 2\hat{\theta}_{MLE} - \frac{1}{B} \sum_{r=1}^B \hat{\theta}_r. \quad (13)$$

This method implies that the estimator in Equation (13) provides a second-order bias correction.

3.2 Simulation Study

In this section, the simulation evaluates the performance of point estimators for the Zeghdoudi distribution using the maximum likelihood, Cox-Snell, and parametric bootstrap methods. The point estimators derived from these three methods are denoted as $\hat{\theta}_{MLE}$, $\hat{\theta}_{CS}$, and $\hat{\theta}_{PB}$, respectively. The effectiveness of the proposed estimators for the Zeghdoudi distribution is evaluated through a Monte Carlo simulation. This simulation study was conducted using R version 4.5.2.

We generated data from the Zeghdoudi distribution using the acceptance-rejection method, with the parameter values (θ) set at 1, 1.5, 2, 2.5, and 3. The sample sizes (n) were set at 10, 20, 30, 40, and 50 to represent small and moderate sample scenarios. For each scenario, the Monte Carlo and bootstrap replication numbers were set to $G = 1,000$ and $B = 1,000$, respectively. The maxLik function in the R software was used to obtain the maximum likelihood estimate for the Zeghdoudi distribution. This study investigates the accuracy of the three estimators $\hat{\theta}$ ($\hat{\theta}_{MLE}$, $\hat{\theta}_{CS}$, $\hat{\theta}_{PB}$) for the parameter θ of the Zeghdoudi distribution using the average bias (AB) and RMSE, as defined in equations 14 and 15, respectively.

$$AB(\hat{\theta}) = \frac{1}{G} \sum_{i=1}^G (\hat{\theta}_i - \theta) \quad (14)$$

$$RMSE(\hat{\theta}) = \sqrt{\frac{1}{G} \sum_{i=1}^G (\hat{\theta}_i - \theta)^2} \quad (15)$$

3.3 Applications to Medical Data

The benefit of this study is that it examines two medical datasets with small or moderate sample sizes, in which the data are positive and right-skewed, and applies point estimators for the Zeghdoudi distribution using maximum likelihood, Cox-Snell, and parametric bootstrap techniques. The first dataset captures the time to infection, measured in months, among patients undergoing kidney dialysis, while the second dataset reports the COVID-19 mortality rate.

To determine whether the two real datasets fit the Zeghdoudi distribution, different criteria are used, including the log-likelihood (log L), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), the Kolmogorov-Smirnov (K-S) test, and the corresponding p-value. Goodness-of-fit measures are applied to assess the effectiveness of the statistical model.

3.3.1 Data Set I

The first dataset contains 28 records of the number of months until infection in kidney dialysis patients, obtained from Klein and Moeschberger (2003). The values in this dataset were transformed to fall between 0 and 1, as described by Al-Omari et al. (2024). The data are as follows:

“0.08333333, 0.11666667, 0.08333333, 0.11666667, 0.48333333, 0.11666667, 0.15, 0.41666667, 0.25, 0.18333333, 0.21666667, 0.21666667, 0.25, 0.91666667, 0.25, 0.28333333, 0.31666667, 0.35, 0.38333333, 0.41666667, 0.45, 0.48333333, 0.71666667, 0.71666667, 0.75, 0.25, 0.75, 0.85.”

3.3.2 Data Set II

The second dataset, as presented by Almongy et al. (2021), details the COVID-19 mortality rate in the Netherlands over a 30-day period. Specifically, this dataset encompasses data collected from 31 March to 30 April 2020. The data are as follows:

“14.918, 10.656, 12.274, 10.289, 10.832, 7.099, 5.928, 13.211, 7.968, 7.584, 5.555, 6.027, 4.097, 3.611,

4.960, 7.498, 6.940, 5.307, 5.048, 2.857, 2.254, 5.431, 4.462, 3.883, 3.461, 3.647, 1.974, 1.273, 1.416, 4.235.”

4. Results

4.1 Results of the Simulation Study

The results in Tables 1 and 2 present the estimated average bias (AB) and root mean squared error (RMSE) for the Zeghdoudi distribution parameter, respectively.

The results in Table 1 show that the absolute values of the average bias for estimators $\hat{\theta}_{CS}$ and $\hat{\theta}_{PB}$ remained lower than those of the maximum likelihood estimator ($\hat{\theta}_{MLE}$) in all scenarios examined. The maximum likelihood estimator ($\hat{\theta}_{MLE}$) appears to have a positive estimated average bias. This estimator yields an overestimate of the true parameter value θ . In most cases, the estimated average biases of all estimators $\hat{\theta}$ ($\hat{\theta}_{MLE}$, $\hat{\theta}_{CS}$, $\hat{\theta}_{PB}$) tend to decrease as sample sizes increase. Based on the results in Table 1, the plots of the average bias for the parameter θ across all estimation techniques are shown in Figure 2.

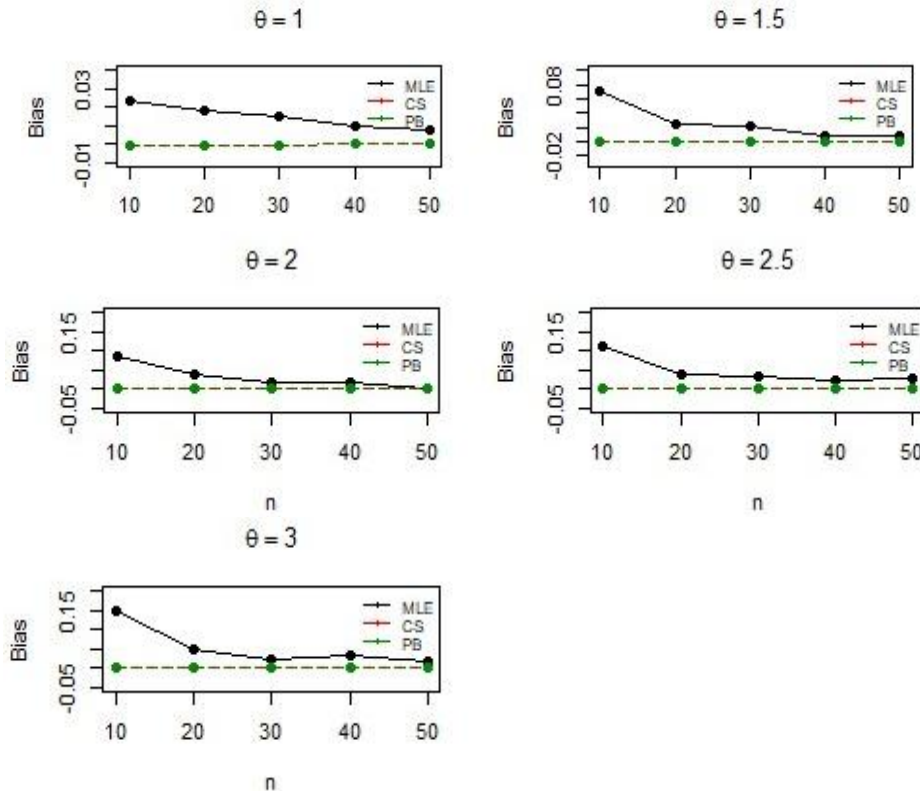


Figure 2 Plot of the average biases (ABs) for θ across all estimation techniques

Table 1 Estimated average bias of the Zeghdoudi distribution parameter

θ	n	Average Bias of $\hat{\theta}$		
		$\hat{\theta}_{MLE}$	$\hat{\theta}_{CS}$	$\hat{\theta}_{PB}$
1	10	0.023360	-0.000217	-0.000202
	20	0.018523	-0.000075	-0.000071
	30	0.015336	-0.000061	-0.000057
	40	0.010241	0.000015	0.000016
	50	0.007623	0.000099	0.000096
1.5	10	0.072161	-0.000432	-0.000421
	20	0.024819	0.000107	0.000092
	30	0.022120	-0.000147	-0.000139
	40	0.009203	0.000050	0.000062
	50	0.006787	0.000051	0.000059
2	10	0.085045	-0.000231	-0.000274
	20	0.038406	-0.000221	-0.000256
	30	0.018125	-0.000240	-0.000257
	40	0.019959	0.000070	0.000073
	50	0.002012	0.000143	0.000147
2.5	10	0.114380	0.000491	0.000473
	20	0.040598	-0.000280	-0.000278
	30	0.035348	0.000120	0.000139
	40	0.023878	0.000218	0.000192
	50	0.027246	-0.000198	-0.000210
3	10	0.149129	-0.000210	-0.000182
	20	0.047119	-0.000063	-0.000047
	30	0.023847	-0.000109	-0.000109
	40	0.035132	-0.000045	-0.000084
	50	0.019508	-0.000108	-0.000113

Table 2 Estimated root mean squared error of the Zeghdoudi distribution parameter

θ	n	RMSE of $\hat{\theta}$		
		$\hat{\theta}_{MLE}$	$\hat{\theta}_{CS}$	$\hat{\theta}_{PB}$
1	10	0.195297	0.006863	0.006387
	20	0.128627	0.002372	0.002254
	30	0.112525	0.001940	0.001815
	40	0.097696	0.000469	0.000500
	50	0.085202	0.003142	0.003030
1.5	10	0.315726	0.013669	0.013321
	20	0.197378	0.003383	0.002919
	30	0.171993	0.004664	0.004398
	40	0.151080	0.001595	0.001949
	50	0.127522	0.001610	0.001849
2	10	0.426636	0.007299	0.008670
	20	0.284523	0.006982	0.008089
	30	0.234122	0.007577	0.008112
	40	0.194064	0.002205	0.002318
	50	0.177067	0.004507	0.004644
2.5	10	0.549332	0.015522	0.014961
	20	0.350042	0.008865	0.008796
	30	0.292768	0.003803	0.004381
	40	0.247893	0.006893	0.006071
	50	0.223477	0.006274	0.006643
3	10	0.662652	0.006647	0.005764
	20	0.440150	0.001989	0.001496
	30	0.331250	0.003462	0.003435
	40	0.295801	0.001418	0.002655
	50	0.264415	0.003418	0.003559

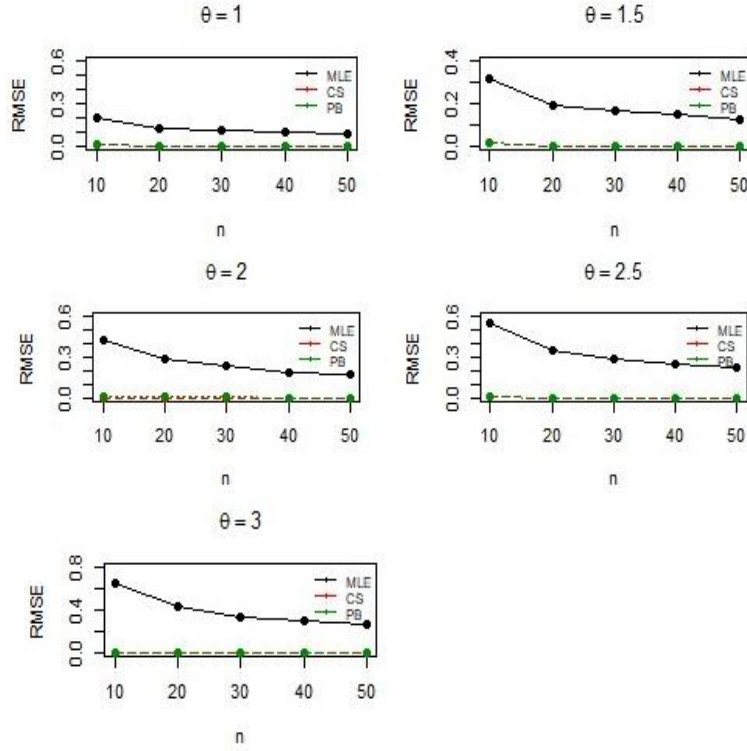


Figure 3 Plot of the root mean squared errors (RMSEs) for θ across all estimation techniques

The results in Table 2 demonstrate that the proposed estimators, $\hat{\theta}_{CS}$ and $\hat{\theta}_{PB}$, have lower RMSEs than the maximum likelihood estimator ($\hat{\theta}_{MLE}$) across all situations. Notably, the parametric bootstrap estimator ($\hat{\theta}_{PB}$) achieves the smallest RMSEs in several cases. All examined estimators show reduced RMSEs as sample sizes increase. The plots of the RMSEs for the parameter θ , shown for all estimation techniques in Figure 3, correspond to the results in Table 2.

As shown in Figure 3, the RMSE of all three estimators declines as the sample size increases, and the curves for the Cox-Snell and parametric bootstrap estimators lie below that of the maximum likelihood estimator across every sample size, with the gap becoming more pronounced as the parameter value grows. This visually confirms the consistency of the estimators and the finite-sample gains of the bias-corrected estimators reported in Table 2.

Tables 1 and 2 show that the proposed Zeghdoudi distribution estimators, which use two bias-reduction methods, outperform the maximum likelihood estimator (MLE) in terms of average bias and root mean squared error (RMSE), as these estimators reduce the bias of the MLE from first order ($O(n^{-1})$) to second order ($O(n^{-2})$) magnitude. All three

estimators are consistent. As the sample size increases, both the average bias and RMSE decrease.

4.2 Results of the Applications to Medical Data

The results for the two medical datasets in Tables 3 and 4 report some basic descriptive measures. The maximum likelihood estimates, their standard errors (SEs), and the goodness-of-fit measures for the two datasets are presented in Tables 5 and 6. This evidence shows that the Zeghdoudi distribution fits both datasets. The estimated parameter values of the Zeghdoudi distribution, derived from three different estimators for the two real datasets, are shown in Tables 7 and 8. We observed that the maximum likelihood estimators ($\hat{\theta}_{MLE}$) for the two datasets tend to overestimate the true value of the parameter θ more than the other estimators. These cases imply that the estimators $\hat{\theta}_{CS}$ and $\hat{\theta}_{PB}$ are more efficient than the maximum likelihood estimator ($\hat{\theta}_{MLE}$), which is consistent with the simulation results. The estimated density functions for the ZD, calculated using all estimator values from both datasets presented in Tables 7 and 8, are illustrated in Figures 4 and 5.

Tables 3 and 4 show that both datasets are positive and right-skewed, with the mean exceeding the median (0.377 versus 0.300 for the dialysis data and 6.157 versus 5.369 for the COVID-19 data) and positive skewness coefficients of 0.765 and 0.834,

respectively. This right-skewed, non-negative behavior is consistent with the shape of the Zeghdoudi distribution and supports its use as a candidate model for both datasets.

The goodness-of-fit results in Tables 5 and 6 confirm this choice. For both datasets, the Kolmogorov-Smirnov test does not reject the Zeghdoudi distribution: the K-S statistics are small (0.109 and 0.093) with large p-values (0.893 and 0.936), indicating no significant departure between the empirical and fitted distributions. Together with the log-likelihood, AIC, and BIC values, these results indicate that the Zeghdoudi distribution provides an adequate fit for both datasets.

The parameter estimates in Tables 7 and 8 show that both bias-corrected estimators adjust the maximum likelihood estimate downward. For the dialysis data, the estimate decreases from 5.965 for the

MLE to 5.873 for the Cox-Snell estimator and 5.845 for the parametric bootstrap, corresponding to reductions of approximately 1.5% and 2.0%. The same downward adjustment is observed for the COVID-19 data, where the MLE of 0.457 decreases to 0.452 and 0.453. Because the simulation study showed that the MLE has a positive bias that overestimates the parameter, these consistent downward corrections on real data are in the direction expected and reinforce the simulation findings. The two corrections are very close to each other, and their ordering differs between the datasets, which is consistent with the simulation result that the Cox-Snell and parametric bootstrap estimators are nearly indistinguishable in finite samples. As the true parameter is unknown for real data, these estimates indicate the direction and magnitude of the correction rather than which estimator is exact.

Table 3 Descriptive measures for the kidney dialysis patients' data

Minimum	Median	Mean	Maximum	SD	Skewness	Kurtosis
0.083	0.300	0.377	0.917	0.247	0.765	2.422

Table 4 Descriptive measures of the COVID-19 death rate in the Netherlands

Minimum	Median	Mean	Maximum	SD	Skewness	Kurtosis
1.273	5.369	6.157	14.918	3.533	0.834	2.953

Table 5 Maximum likelihood estimate, standard error, and goodness-of-fit measures for the kidney dialysis patients' data

Distribution	Estimate (SE)	Log L	AIC	BIC	K-S	p-value (K-S)
ZD	5.965 (0.722)	3.913	-5.826	-4.494	0.109	0.893

Table 6 Maximum likelihood estimate, standard error, and goodness-of-fit measures for the COVID-19 death rate in the Netherlands

Distribution	Estimate (SE)	Log L	AIC	BIC	K-S	p-value (K-S)
ZD	0.457 (0.048)	-76.928	155.857	157.258	0.093	0.936

Table 7 Parameter estimates of the ZD for the kidney dialysis patients' data

Estimators	θ
$\hat{\theta}_{MLE}$	5.965
$\hat{\theta}_{CS}$	5.873
$\hat{\theta}_{PB}$	5.845

Table 8 Parameter estimates of the ZD for the COVID-19 death rate in the Netherlands

Estimators	θ
$\hat{\theta}_{MLE}$	0.457
$\hat{\theta}_{CS}$	0.452
$\hat{\theta}_{PB}$	0.453

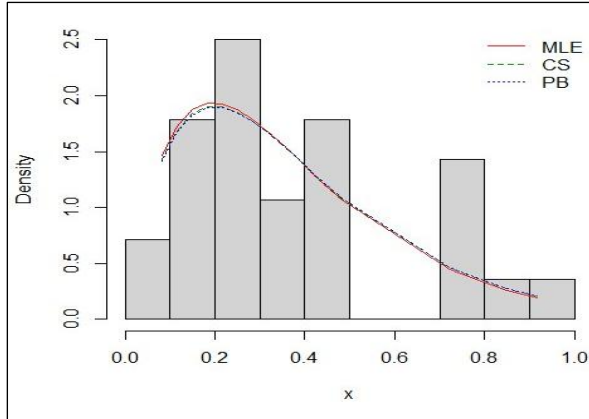


Figure 4 Estimated density functions for the first dataset

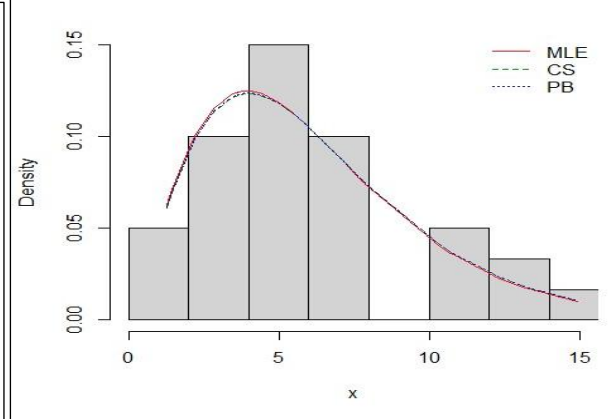


Figure 5 Estimated density functions for the second dataset

Figures 4 and 5 overlay the fitted Zeghdoudi densities, obtained from each of the three estimates, on the histograms of the observed data. The three density curves are nearly coincident, reflecting the small differences among the estimates in Tables 7 and 8, and all three follow the right-skewed shape of the data. This visually confirms that the bias correction refines the parameter estimate without distorting the fit of the distribution.

5. Conclusions

This research presents point estimators constructed using two bias-correction methods, namely the Cox-Snell method and the parametric bootstrap method, for estimating the parameter of the Zeghdoudi distribution in small and moderate sized samples. The performance of these proposed estimators, compared with that of the classical maximum likelihood estimator, is evaluated through simulation studies under diverse scenarios.

The results were analyzed to identify the most efficient estimator, using average bias (AB) and root mean squared error (RMSE) as criteria. The overall study results demonstrated that the point estimators derived from the two bias-correction techniques performed better than the classical maximum likelihood estimator in terms of average bias and root mean squared error, as observed in both numerical and medical dataset analyses. Notably, these proposed estimators improved accuracy and enhanced the reliability of parameter estimation by reducing bias to the second order. Furthermore, the point estimator constructed using the parametric bootstrap method outperformed both the maximum likelihood and Cox-Snell methods in various situations. Our study recommends the two bias-corrected estimators for

estimating the parameter of the Zeghdoudi distribution, especially for small to moderate sample sizes.

This study focuses on the Zeghdoudi distribution, a lifetime distribution applied in the medical field, specifically for small and moderate samples where the data are right-skewed. In general, numerous lifetime distributions have been developed for fitting real-world data in various fields. Estimating the parameters of any lifetime distribution using the classical maximum likelihood method can be problematic, especially when the data have small sample sizes. Further research could extend this approach to other lifetime distributions applied to small to moderate sample sizes, enhancing the accuracy of parameter estimation by using the bias-corrected approaches presented in this paper.

This study has some limitations. It addresses a single-parameter distribution and a single parameter θ ; the parametric bootstrap is computationally more demanding than the closed-form Cox-Snell correction; and the findings are specific to the ZD. In practice, we therefore recommend the Cox-Snell estimator when a fast, closed-form correction is preferred, and the parametric bootstrap when minimizing RMSE is the priority and additional computation is acceptable. Further research could extend these bias-corrected approaches to other lifetime distributions and to multi-parameter models applied to small and moderate samples.

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7. Abbreviations

Abbreviation	Full Term
AB	Average Bias
AIC	Akaike Information Criterion
BIC	Bayesian Information Criterion
CDF	Cumulative Distribution Function
CS	Cox-Snell
K-S	Kolmogorov-Smirnov
MLE	Maximum Likelihood Estimator
PB	Parametric Bootstrap
PDF	Probability Density Function
RMSE	Root Mean Square Error
SD	Standard Deviation
SE	Standard Error
ZD	Zeghdoudi Distribution

8. CRediT Statement

Chanakan Sungboonchoo: Conceptualization, Methodology, Software, Validation, Formal Analysis, Investigation, Resources, Data Curation, Writing – Original Draft, Writing – Review & Editing, Visualization.

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