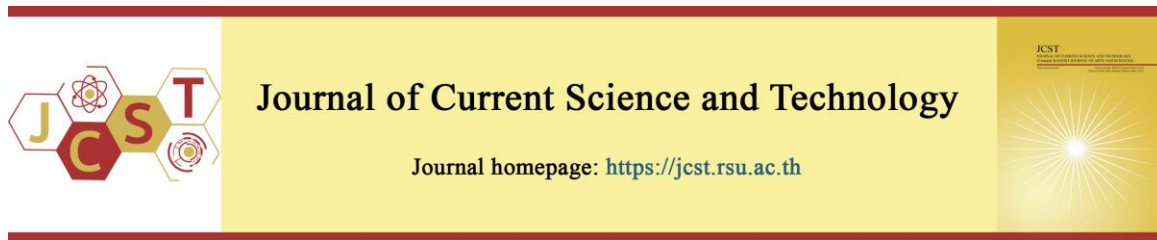


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Impact of Modeling Simplifications on the Dynamics and Control of the Furuta Pendulum

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Abstract

This study analyzes the impact of modeling simplifications on the dynamics and control performance of the Furuta pendulum. A complete multibond graph model is developed, incorporating full system dynamics, and providing an energetically consistent and modular framework. The model is validated against a Simulink-Simscape reference, achieving a maximum normalized root mean squared error (NRMSE) of 0.2035×10^{-3} . The commonly used simplified model, which neglects secondary inertias, is compared with full-dynamics models under open- and closed-loop conditions. Open-loop results show appreciable discrepancies, with NRMSE values increasing notably for both θ_1 and θ_2 as the inertias are progressively increased. A nonlinear control scheme based on energy shaping, collocated partial feedback linearization, and linear quadratic regulation (LQR) is designed using the simplified model. In closed loop, the controller achieves swing-up and stabilization in all cases within 10 seconds. For nominal and moderately increased inertias, performance degradation is minimal, with settling times around 7.4 s and control effort between 0.453 and 0.475 N²m²s. However, for large inertias, the settling time increases to 8.91 s, control effort rises to 0.655 N²m²s, and an additional oscillation is required. These results show that simplified models are suitable for control design under typical conditions, while full-dynamics models are essential for validation and robustness assessment.

Keywords: furuta pendulum, rotary inverted pendulum, bond graph, multibond graph, energy shaping, linear-quadratic regulator (LQR)

1. Introduction

The Furuta pendulum is a nonlinear mechanical system consisting of a horizontal rotating arm and a pendulum attached to its end, which oscillates in a plane perpendicular to the arm (Figure 1). It has characteristics that make it particularly interesting in the field of automatic control: it is nonlinear, has unstable equilibrium points, and has fast dynamics, while also being relatively simple to implement. Various control techniques have been used to balance the Furuta pendulum, from classical approaches such as proportional–integral–derivative (PID) control and linear quadratic regulator (LQR) control to more advanced methods such as sliding mode control,

adaptive control, and strategies based on artificial intelligence (Acosta, 2010; De Jesús Rubio, 2018; Hamza et al., 2019; Moreno-Valenzuela et al., 2016; Turchetta et al., 2020; Yang & Zheng, 2018; Zabihiyar et al., 2020).

Although the arm has planar motion, the pendulum has three-dimensional motion. Many authors only consider the inertia of the pendulum in the direction parallel to the arm, ignoring the other two components (Fantoni & Lozano, 2002; Furuta et al., 1991; Olfati-Saber, 1999; Rodríguez-Cortés, 2019; Xin & Liu, 2014). Other authors consider the pendulum as a slender bar with equal inertia in two axes, but ignoring the inertia in the longitudinal

direction of the pendulum (Iwase et al., 2006). Less common is the complete omission of the pendulum's inertia, considering it as a point mass (Egeland & Gravdahl, 2002). Only two studies were found that include the complete inertia of the pendulum, but discrepancies exist between the two models (Awtar et al., 2002; Cazzolato & Prime, 2011).

Although simulations have shown that simplifying the inertia of the pendulum affects its open-loop dynamics (Cazzolato & Prime, 2011), the extent to which this simplification affects closed-loop dynamics when a control system is included has not yet been quantified. This is important because almost all proposed control systems are tested on simplified models.

The bond graph methodology is a unified tool for the physical modeling of dynamic systems, explicitly representing the exchange of energy between its components. Traditional modeling methods, such as the Euler–Lagrange approach, present some limitations in the modeling of mechanical systems that are not encountered in bond graph formulations. In particular, they do not inherently enforce energy conservation, requiring careful verification during derivation. The inclusion of dissipative effects and non-conservative interactions can also become unsystematic and error-prone. Moreover, its extension to multi-domain systems (e.g., electromechanical couplings) is less straightforward and often demands additional formulations. As system complexity increases, the resulting equations may become algebraically cumbersome and less transparent. In contrast, bond graph modeling provides a modular, energy-based representation that naturally preserves physical consistency and facilitates subsystem interconnection (Karnopp et al., 2012). In particular,

the multibond graph extension is especially suitable for modeling three-dimensional multibody systems, as it allows the multiple power interactions that occur in complex spatial movements to be represented in a compact and structured way (Gonzalez-Avalos et al., 2023; Maini & Vaz, 2020). This technique facilitates the creation of energetically consistent models, preserving the physical structure of the system and simplifying the subsequent analysis and design of control strategies (Borutzky, 2010).

This study addresses the problem of quantifying the extent to which controllers designed using simplified models retain their performance when evaluated on more accurate, full-dynamics models. The results demonstrate that control strategies developed based on the widely adopted simplified model of the Furuta pendulum remain effective when applied to the full-dynamics model, despite moderate discrepancies in the system dynamics. Specifically, an aggressive nonlinear control strategy for swing-up and stabilization based on Energy Shaping, Collocated Partial Feedback Linearization (CPFL), and Linear Quadratic Regulator (LQR) is designed using the simplified model and subsequently evaluated on both the simplified and the full-dynamics model.

Additionally, this study presents a bond graph model of the Furuta pendulum that is modular, energetically consistent, incorporates the full system dynamics, and can be readily reused in more complex configurations. Furthermore, the models from the only two studies in literature that include the full dynamics are examined and contrasted, showing that one is accurate while the other exhibits minor discrepancies. The open-loop responses of both models under a step input are also compared.

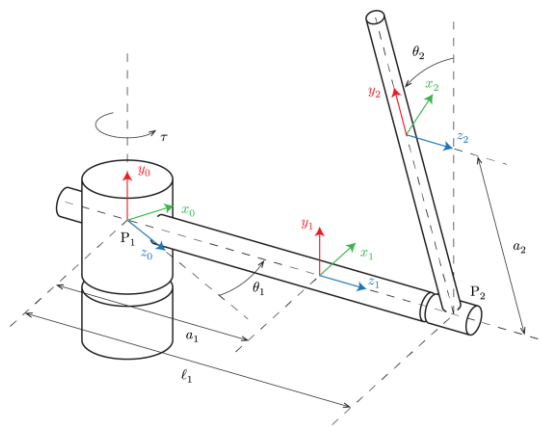


Figure 1 The Furuta Pendulum

2. Objectives

The main objective of this study is to quantify whether controllers designed based on simplified models of the Furuta pendulum maintain satisfactory performance when applied to more accurate, full-dynamics models. To achieve this, the following specific objectives are defined:

1. Develop an energetically consistent model that captures the full dynamics of the Furuta pendulum, for which the bond graph methodology is adopted.
2. Analyze and identify discrepancies in previously reported models that claim to represent the exact dynamics of the Furuta pendulum.
3. Design control strategies for both swing-up and stabilization of the Furuta pendulum using energy shaping, partial feedback linearization, and LQR.
4. Evaluate, through numerical simulations, the degradation in controller performance and the changes in system dynamics when transitioning from the simplified model to the full-dynamics model.

3. Methods

3.1 Bond Graph Modeling

The Furuta pendulum is shown in Figure 1. The arm is actuated by a torque τ while the pendulum joint is free to rotate. The following assumptions are made during the modeling process:

1. The rotary arm and pendulum are modeled as rigid bodies, neglecting elastic deformations.
2. The mass distribution and inertial parameters are constant and known.
3. The pendulum is described using a full inertia tensor.
4. The system uses ideal revolute joints, without backlash, clearance, or compliance.
5. Only viscous friction is considered; other friction effects are neglected.
6. The model assumes no external disturbances, such as aerodynamic effects or external forces.
7. The actuator dynamics are neglected, and torque is applied directly without delay or saturation.

The following three reference frames are defined: the inertial frame $\{0\}$, the frame $\{1\}$ fixed to the center of mass of the arm, and the frame $\{2\}$ fixed to the center of mass of the pendulum. In Figure 1, the hinge points P_1 and P_2 are also defined, which are

useful for the bond graph modeling process. The joint variables that define the configuration of the system are θ_1 and θ_2 .

The multibond graph model of the Furuta pendulum is built based on the model proposed by Bos (1986) for a rigid body (see Figure 2). In this model, a 1-junction represents the absolute angular velocity of the body, expressed in its own reference frame. External torques, an inertial element, and the Eulerian Junction Structure (EJS) represented by the symbol MGY are connected to this 1-junction. The external torques also include those produced by external forces. To add them, the velocities are related by the well-known kinematic relationship $v_j = v_c + \omega \times x_j$, where v_c is the velocity of the center of mass, v_j is the velocity of point P_j , ω is the angular velocity, and x_j is the position vector from the center of mass to point P_j . Using column vectors, the above equation becomes $v_c^i = v_j^i + \tilde{x}_j^i \omega_i^i$, where the index i indicates that all elements are represented according to the frame of body i itself, and $\tilde{x}_j^i$ is the skew-symmetric matrix of x_j^i . This kinematic constraint is implemented in the bond graph model by means of a constant modulus transformer.

While angular velocities and torques are expressed in the body-fixed reference frame, the linear velocity of the center of mass is represented in the base reference frame. The absolute linear velocity of the body's center of mass in the base frame, denoted as v_c^0 , is depicted at the bottom of Figure 2 by means of a 1-junction, to which the gravitational force and the body mass are connected.

To interconnect the bodies that constitute a multibody system, predefined joint models can be employed, such as the one illustrated in Figure 3, which corresponds to a revolute joint. In this representation, the multibond is split into one-dimensional bonds and a 0-junction is inserted into the bond representing the angular velocity in the direction allowed by the revolute joint. Friction effects can be incorporated through a resistive element, and an external torque can be applied if the joint is actuated. When rigid bodies are interconnected through joints, it is necessary to transform velocity variables from the reference frame of one body to that of the adjacent body (Tienego & Bos, 1985).

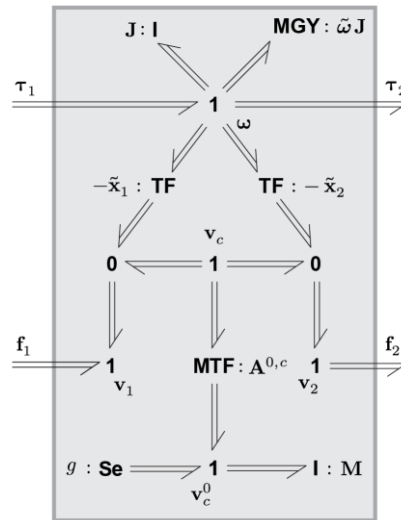


Figure 2 Multibond graph model of three-dimensional body

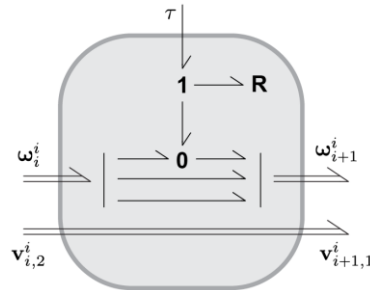


Figure 3 Multibond graph model of a revolute joint

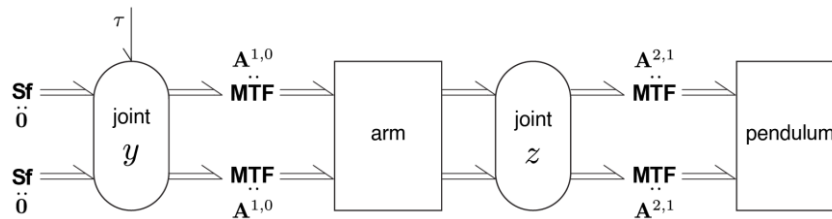


Figure 4 General multibond graph model of the Furuta pendulum

Based on the rigid-body and revolute joint models, the multibond graph representation of the Furuta pendulum can be constructed, as illustrated in Figure 4. Rigid bodies are depicted as rectangular elements, while joints are represented by pill-shaped elements. The first joint is actuated and rotates about the y -axis, whereas the second joint is unactuated and rotates about the z -axis. On the left-hand side, zero-valued flow sources represent the null velocity of the inertial base frame.

By replacing, in the previous model, the detailed representations of the rigid-body and joint models, and after performing the corresponding simplifications, the detailed model shown in Figure 5 is obtained. The points at which forces are exerted, both for the arm and for the pendulum, are the hinge points P_1 and P_2 . It can be shown that the Eulerian Junction Structure corresponding to the arm does not contribute any torque, so it can be omitted from the model.

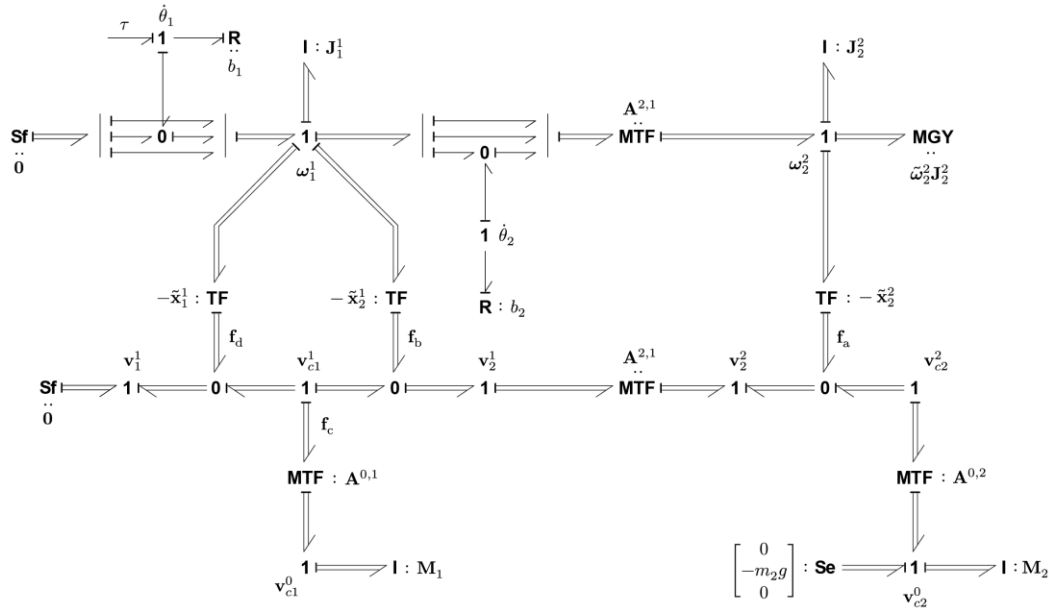


Figure 5 Detailed multibond graph model of the Furuta pendulum

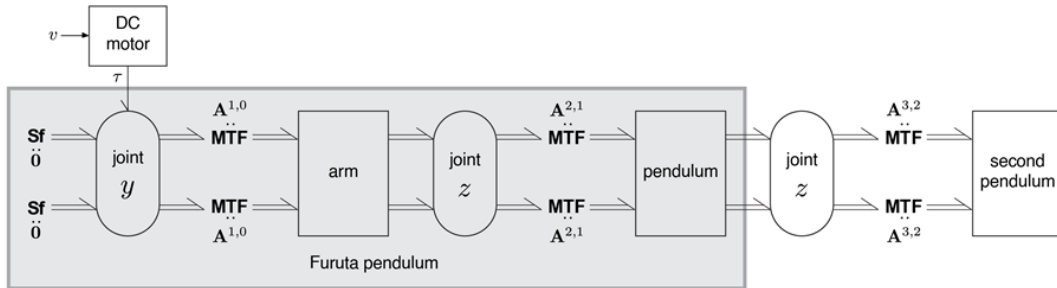


Figure 6 Multibond graph model of the double-link rotary inverted pendulum based on the Furuta pendulum

Some of the most recognized features of the bond graph methodology are its modularity, scalability, and multidomain integration capability. Bond graph models can be systematically assembled from reusable modules, such as those previously described for rigid bodies and revolute joints. This modular structure enables straightforward scaling of the previously presented Furuta pendulum model to more complex configurations, such as double- and triple-link rotary inverted pendulums. It also facilitates the seamless integration of actuator models, for instance, an electric motor. In contrast, when conventional approaches such as the Euler–Lagrange formulation are employed, the modeling process often needs to be redeveloped almost from scratch when extending the system. Furthermore, bond graph models can be directly simulated in software environments such as 20-sim without the explicit derivation of mathematical equations. As an

illustration, Figure 6 presents the multibond graph model of a double-link rotary inverted pendulum actuated by an electric motor, which is built upon the previously introduced Furuta pendulum model.

3.2 Dynamic Model

Based on the bond graph model shown in Figure 5, the dynamic mathematical model of the Furuta pendulum is derived. To this end, the necessary kinematic relationships between linear and angular velocities are first established. The rotation matrix of frame $\{j\}$ with respect to frame $\{i\}$ is denoted as $A^{i,j}$. Thus,

$$A^{0,1} = \text{Rot}_y(\theta_1) = \begin{bmatrix} \cos \theta_1 & 0 & \sin \theta_1 \\ 0 & 1 & 0 \\ -\sin \theta_1 & 0 & \cos \theta_1 \end{bmatrix} \quad (1)$$

$$A^{1,2} = \text{Rot}_z(\theta_2) = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2)$$

Moreover, $A^{0,2} = A^{0,1} A^{1,2}$. The position of the hinge point P_i expressed with respect to frame $\{j\}$ is denoted as x_i^j . Hence, we have:

$$\begin{cases} x_1^1 = [0 & 0 & -a_1]^T \\ x_2^1 = [0 & 0 & l_1 - a_1]^T \\ x_2^2 = [0 & -a_2 & 0]^T \end{cases} \quad (3)$$

The absolute angular velocity of body i , expressed in the coordinates of the body itself, is denoted as ω_i^i . These velocities for both bodies are:

$$\omega_1^1 = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \quad (4)$$

$$\omega_2^2 = A^{2,1} \left(\omega_1^1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} \right) = A^{2,1} \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \sin(\theta_2) \dot{\theta}_1 \\ \cos(\theta_2) \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \quad (5)$$

The following kinematic relationships are directly derived from the bond graph, according to the causality indicated in Figure 5:

$$\begin{cases} v_1^1 = 0 \\ v_{c1}^1 = v_1^1 + \tilde{x}_1^1 \omega_1^1 \\ v_{c1}^0 = A^{0,1} v_{c1}^1 \\ v_2^1 = v_{c1}^1 - \tilde{x}_2^1 \omega_1^1 \\ v_2^2 = A^{2,1} v_2^1 \\ v_{c2}^2 = v_2^2 + \tilde{x}_2^2 \omega_2^2 \\ v_{c2}^0 = A^{0,2} v_{c2}^2 \end{cases} \quad (6)$$

Some intermediate efforts are computed, which will subsequently be used to derive the dynamic equations. These efforts are indicated in Figure 5 and are as follows:

$$\begin{cases} f_a = -A^{2,0} \left(M_2 \dot{v}_{c2}^0 - \begin{bmatrix} 0 \\ -m_2 g \end{bmatrix} \right) \\ f_b = -A^{1,2} f_a \\ f_c = A^{1,0} M_1 \dot{v}_{c1}^0 \\ f_d = -f_b - f_c \end{cases} \quad (7)$$

Next, the torques applied to each body are analyzed. The summation of torques at the 1-junction associated with ω_2^2 yields:

$$\tau_2 = A^{1,2} (J_2^2 \dot{\omega}_2^2 + \tilde{\omega}_2^2 J_2^2 \omega_2^2 + \tilde{x}_2^2 f_a) \quad (8)$$

If we write $\tau_2 = [\tau_{2x} \ \tau_{2y} \ \tau_{2z}]^T$, the summation of torques at the 1-junction associated with $\dot{\theta}_2$ gives:

$$0 = \tau_{2z} + b_2 \dot{\theta}_2 \quad (9)$$

On the other hand, the summation of torques at the 1-junction associated with ω_1^1 results in:

$$\tau_1 = J_1^1 \dot{\omega}_1^1 + \tilde{x}_2^1 f_b + \tilde{x}_1^1 f_d + \tau_2 \quad (10)$$

If we write $\tau_1 = [\tau_{1x} \ \tau_{1y} \ \tau_{1z}]^T$, the summation of torques at the 1-junction associated with $\dot{\theta}_1$ yields:

$$\tau = \tau_{1y} + b_1 \dot{\theta}_1 \quad (11)$$

Using MATLAB's symbolic manipulation toolbox, Equations (9) and (11) result in:

$$\begin{aligned} \tau &= (m_1 a_1^2 + m_2 l_1^2 + J_{1y} + (m_2 a_2^2 + J_{2x}) \sin^2(\theta_2) + J_{2y} \cos^2(\theta_2)) \ddot{\theta}_1 \\ &\quad - m_2 a_2 l_1 \cos(\theta_2) \ddot{\theta}_2 + (m_2 a_2 l_1 \sin(\theta_2)) \dot{\theta}_2^2 \\ &\quad + (m_2 a_2^2 + J_{2x} - J_{2y}) \sin(2\theta_2) \dot{\theta}_1 \dot{\theta}_2 + b_1 \dot{\theta}_1 \\ 0 &= -m_2 a_2 l_1 \cos(\theta_2) \ddot{\theta}_1 + (m_2 a_2^2 + J_{2z}) \ddot{\theta}_2 \\ &\quad + \frac{1}{2} (-m_2 a_2^2 - J_{2x} + J_{2y}) \sin(2\theta_2) \dot{\theta}_1^2 \\ &\quad - a_2 g m_2 \sin(\theta_2) + b_2 \dot{\theta}_2 \end{aligned} \quad (12)$$

The equations in (12) represent the inverse dynamic model of the Furuta pendulum. These can be written compactly as:

$$T = M(q) \ddot{q} + C(q, \dot{q}) \dot{q} + G(q) + F(\dot{q}) \quad (13)$$

Where $q = [\theta_1 \ \theta_2]^T$ is the vector of joint variables and $T = [\tau \ 0]^T$ is the vector of external torques. The matrices in (13) are:

$$\begin{aligned} M(q) &= \begin{bmatrix} m_1 a_1^2 + m_2 l_1^2 + J_{1y} & + (m_2 a_2^2 + J_{2x}) \sin^2(\theta_2) - m_2 a_2 l_1 \cos(\theta_2) \\ + J_{2y} \cos^2(\theta_2) & - m_2 a_2 l_1 \cos(\theta_2) \\ - m_2 a_2 l_1 \cos(\theta_2) & m_2 a_2^2 + J_{2z} \end{bmatrix} \\ C(q, \dot{q}) &= \begin{bmatrix} (m_2 a_2^2 + J_{2x} - J_{2y}) \sin(2\theta_2) \dot{\theta}_2 & m_2 a_2 l_1 \sin(\theta_2) \dot{\theta}_2 \\ \frac{1}{2} (-m_2 a_2^2 - J_{2x} + J_{2y}) \sin(2\theta_2) \dot{\theta}_1 & 0 \end{bmatrix} \\ G(q) &= \begin{bmatrix} 0 \\ -a_2 g m_2 \sin(\theta_2) \end{bmatrix} \\ F(\dot{q}) &= \begin{bmatrix} b_1 \dot{\theta}_1 \\ b_2 \dot{\theta}_2 \end{bmatrix} \end{aligned} \quad (14)$$

Only two papers were found that include the complete inertia of the pendulum. The dynamic model reported by Cazzolato and Prime (2011) is identical to the one derived in this work. However, the model presented by Awtar et al. (2002) contains minor errors. A comparison with (12) reveals a sign error in the term $m_2 a_2 l_1 \sin(\theta_2) \dot{\theta}_2^2$, as well as a typographical error in the term $(m_2 a_2^2 + J_{2x} - J_{2y}) \sin(2\theta_2) \dot{\theta}_1 \dot{\theta}_2$. For a proper comparison, it should be noted that the angle θ_2 was defined differently, and the moments of inertia were expressed with respect to axes in a different order.

Based on the previously derived dynamic model, the most commonly used model can be derived. This model results from assuming that the pendulum's moments of inertia about the x and y axes are negligible. The moment of inertia J_{2y} is very small compared to J_{2z} , and thus neglecting it may be reasonable; however, J_{2x} is very similar or even identical to J_{2z} , and cannot be neglected without further analysis. In well-known studies, the dynamic equations are derived using the Lagrangian formulation under the assumption that the rotational kinetic energy of the pendulum is solely $J_{2z} \dot{\theta}_2^2/2$, thereby ignoring the contributions associated with the inertia terms J_{2x} and J_{2y} (Fantoni & Lozano, 2002; Furuta et al., 1991; Olfati-Saber, 1999; Ramírez-Neria et al., 2014; Xin & Liu, 2014). In this case, the matrices $M(q)$ and $C(q, \dot{q})$ become

$$M(q) = \begin{bmatrix} m_1 a_1^2 + m_2 l_1^2 + J_{1y} & -m_2 a_2 l_1 \cos(\theta_2) \\ +m_2 a_2^2 \sin^2(\theta_2) & \\ -m_2 a_2 l_1 \cos(\theta_2) & m_2 a_2^2 + J_{2z} \end{bmatrix} \quad (15)$$

$$C(q, \dot{q}) = \begin{bmatrix} m_2 a_2^2 \sin(2\theta_2) \dot{\theta}_2 & m_2 a_2 l_1 \sin(\theta_2) \dot{\theta}_2 \\ -\frac{1}{2} m_2 a_2^2 \sin(2\theta_2) \dot{\theta}_1 & 0 \end{bmatrix}$$

whereas the matrices $G(q)$ and $F(\dot{q})$ remain identical.

While control engineers frequently rely on simplified models to synthesize control laws, intentionally abstracting reality to keep the mathematics tractable, this approach inherently depends on the controller's ability to handle unmodeled dynamics. However, to guarantee that a controller is truly capable of managing these structural inaccuracies without compromising closed-loop stability, a highly precise baseline model is indispensable. The exact multibond graph formulation developed herein acts as a rigorous virtual testbed for validation. It allows us to explicitly quantify the boundaries of unmodeled dynamics and verify

whether a controller designed under simplified assumptions survives the transition to true physical complexity. The following section details the design of a swing-up and stabilization controller based on the simplified model, whose performance will subsequently be stress-tested against the exact full-dynamics model.

3.3 Swing-up and Stabilization

An energy-based controller has been developed to drive the pendulum from the downward vertical position to the upright equilibrium. This energy-based controller is subsequently switched to an LQR controller when the system state approaches the vicinity of the upright equilibrium point. Both the energy-based controller and the LQR were designed based on a partially linearized model obtained via collocated partial feedback linearization (CPFL). The entire control design is founded on the most commonly used model described in (15).

3.3.1 Collocated Partial Feedback Linearization

Collocated partial feedback linearization is employed in underactuated systems to linearize the dynamics associated with the actuated coordinate while preserving the nonlinearity of the unactuated internal dynamics (Spong et al., 2020). This approach is particularly useful in energy-based strategies, such as swing-up control.

The dynamic model of the system can be decomposed as:

$$\begin{aligned} M_{11} \ddot{\theta}_1 + M_{12} \ddot{\theta}_2 + C_{11} \dot{\theta}_1 + C_{12} \dot{\theta}_2 + G_1 + F_1 &= \tau \\ M_{21} \ddot{\theta}_1 + M_{22} \ddot{\theta}_2 + C_{21} \dot{\theta}_1 + C_{22} \dot{\theta}_2 + G_2 + F_2 &= 0 \end{aligned} \quad (16)$$

By solving for $\ddot{\theta}_2$ from the second equation and substituting it into the first, we obtain:

$$\bar{M}_{11} \ddot{\theta}_1 + \bar{C}_1 + \bar{G}_1 + \bar{F}_1 = \tau \quad (17)$$

where

$$\begin{aligned} \bar{M}_{11} &= M_{11} - \frac{M_{12} M_{21}}{M_{22}} \\ \bar{C}_1 &= C_{11} \dot{\theta}_1 + C_{12} \dot{\theta}_2 - \frac{M_{12}}{M_{22}} C_{21} \dot{\theta}_1 - \frac{M_{12}}{M_{22}} C_{22} \dot{\theta}_2 \\ \bar{G}_1 &= G_1 - \frac{M_{12}}{M_{22}} G_2 \\ \bar{F}_1 &= F_1 - \frac{M_{12}}{M_{22}} F_2. \end{aligned} \quad (18)$$

Consider the following control law:

$$\tau = \bar{M}_{11} u + \bar{C}_1 + \bar{G}_1 + \bar{F}_1 \quad (19)$$

where u is a new input, the resulting system dynamics are:

$$\begin{aligned}\ddot{\theta}_1 &= u \\ M_{22}\ddot{\theta}_2 + C_{21}\dot{\theta}_1 + C_{22}\dot{\theta}_2 + G_2 + F_2 &= -M_{21}u\end{aligned}\quad (20)$$

$$\begin{aligned}\text{The second equation can be expanded as} \\ (m_2 a_2^2 + J_{2z})\ddot{\theta}_2 - m_2 a_2^2 \sin(\theta_2) \cos(\theta_2) \dot{\theta}_1^2 + b_2 \dot{\theta}_2 \\ - m_2 g a_2 \sin(\theta_2) = m_2 l_1 a_2 \cos(\theta_2) u\end{aligned}\quad (21)$$

The system is now linear with respect to the actuated variable θ_1 , while retaining the nonlinearity in the unactuated variable θ_2 .

3.3.2 Swing-up via Energy Shaping

Energy shaping consists of designing a control law that explicitly modifies the total or partial energy of the system by injecting or dissipating it to drive the system from the stable equilibrium to the upright unstable equilibrium (Xin & Liu, 2014). The kinetic energy of an inertial element in a multibond graph model is

$$T_1 = \frac{1}{2} \dot{f}^T I f \quad (22)$$

where f is its flow and I is its inertia matrix. The total kinetic energy is the sum of the individual kinetic energy. From the bond graph model, it can be expressed as:

$$T = \frac{1}{2} v_{c1}^T M_1 v_{c1} + \frac{1}{2} v_{c2}^T M_2 v_{c2} + \frac{1}{2} \omega_1^T J_1 \omega_1 + \frac{1}{2} \omega_2^T J_2 \omega_2 \quad (23)$$

The potential energy of the system is due to gravity and can be determined from the bond graph model as:

$$U = -g^T x_{c2}^0 = - \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}^T \int_0^t v_{c2}^0 dt. \quad (24)$$

$$\begin{aligned}\text{The total energy is} \\ E_T = T + U\end{aligned}\quad (25)$$

For energy shaping, one can consider either the total or partial energy. Let us consider the partial energy of the system as that of the pendulum under the condition that the arm is at the desired rest state, i.e., $\dot{\theta}_1 = 0$. This energy is given by:

$$E = \frac{1}{2} (m_2 a_2^2 + J_{2z}) \dot{\theta}_2^2 + m_2 g a_2 \cos(\theta_2). \quad (26)$$

The desired energy is the value of E at the desired state, which is $E_d = m_2 g a_2$. The energy error is defined as:

$$\tilde{E} = E - E_d \quad (27)$$

$$\begin{aligned}\text{The time derivative of this error is given by:} \\ \dot{\tilde{E}} = [(m_2 a_2^2 + J_{2z})\ddot{\theta}_2 - m_2 g a_2 \sin(\theta_2)] \dot{\theta}_2\end{aligned}\quad (28)$$

$$\begin{aligned}\text{By substituting } \ddot{\theta}_2 \text{ from (21), we obtain:} \\ \dot{\tilde{E}} = [m_2 l_1 a_2 \cos(\theta_2) u + m_2 a_2^2 \sin(\theta_2) \cos(\theta_2) \dot{\theta}_1^2 - b_2 \dot{\theta}_2] \dot{\theta}_2\end{aligned}\quad (29)$$

$$\begin{aligned}\text{Choosing } u \text{ such that} \\ \dot{\tilde{E}} = -k_v \cos^2(\theta_2) \dot{\theta}_2^2 \tilde{E}\end{aligned}\quad (30)$$

where $k_v > 0$, ensures that $\tilde{E}$ is non-increasing and converges to zero under appropriate excitation conditions. Equating the right-hand sides of Equations (29) and (30) yields:

$$u = -k_v \frac{\cos(\theta_2) \dot{\theta}_2}{m_2 l_1 a_2} \tilde{E} - \frac{a_2}{l_1} \sin(\theta_2) \dot{\theta}_1^2 + \frac{b_2}{m_2 l_1 a_2 \cos(\theta_2)} \dot{\theta}_2 \quad (31)$$

The control input in Equation (31) can compensate for friction in the pendulum joint when $\cos(\theta_2)$ is not too small. To avoid singularities, a smooth regulation is considered:

$$\frac{1}{\cos(\theta_2)} \rightarrow \frac{\cos(\theta_2)}{\cos^2(\theta_2) + \epsilon} \quad (32)$$

where ϵ is small. It then follows that

$$u = -k_v \frac{\cos(\theta_2) \dot{\theta}_2}{m_2 l_1 a_2} \tilde{E} - \frac{a_2}{l_1} \sin(\theta_2) \dot{\theta}_1^2 + \frac{b_2}{m_2 l_1 a_2} \frac{\cos(\theta_2)}{\cos^2(\theta_2) + \epsilon} \dot{\theta}_2 \quad (33)$$

The control law in Equation (33) is designed to provide the pendulum with the required energy but is not intended to stabilize the arm. For this purpose, a proportional-derivative (PD) controller is applied to the θ_1 variable, so that the resulting new input is finally given by:

$$u = -k_v \frac{\cos(\theta_2) \dot{\theta}_2}{m_2 l_1 a_2} \tilde{E} - \frac{a_2}{l_1} \sin(\theta_2) \dot{\theta}_1^2 + \frac{b_2}{m_2 l_1 a_2} \frac{\cos(\theta_2)}{\cos^2(\theta_2) + \epsilon} \dot{\theta}_2 - k_p \theta_1 - k_D \dot{\theta}_1 \quad (34)$$

3.3.3 Stabilization via LQR

When the system approaches the upright equilibrium point, the controller must switch to one designed to stabilize the system. For this purpose, a linear quadratic regulator based on a partial feedback linearization model was designed. Let the equilibrium point be defined as $q_0 = \dot{q}_0 = \ddot{q}_0 = [0 \ 0]^T$ and $u = 0$. The dynamics described in Equations (20) and (21) are then linearized as:

$$\begin{aligned}\ddot{\theta}_1 &= u \\ \ddot{\theta}_2 &= \frac{m_2 g a_2}{m_2 a_2^2 + J_{2z}} \theta_2 - \frac{b_2}{m_2 a_2^2 + J_{2z}} \dot{\theta}_2 + \frac{m_2 l_1 a_2}{m_2 a_2^2 + J_{2z}} u\end{aligned}\quad (35)$$

The state vector is defined as,

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = [\theta_1 \quad \theta_2 \quad \dot{\theta}_1 \quad \dot{\theta}_2]^T \quad (36)$$

in state-space form, the system can be expressed as $\dot{x} = Ax + Bu$ with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{m_2 g a_2}{m_2 a_2^2 + J_{2z}} & 0 & -\frac{b_2}{m_2 a_2^2 + J_{2z}} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{m_2 l_1 a_2}{m_2 a_2^2 + J_{2z}} \end{bmatrix} \quad (38)$$

In LQR-based stabilization, the control law takes the following form:

$$u = -Kx \quad (39)$$

where $K = [K_1 \ K_2 \ K_3 \ K_4]$ is the gain vector to be determined. In the LQR control approach, the cost functional is defined as $J = \int_0^\infty (x^T Q x + R \tau^2) dt$, which must be minimized. The matrix Q and the scalar R are parameters that describe the relative weighting of the state variables and the control input. The gain vector that minimizes the cost functional is given by $K = R^{-1} B^T P$, where P results from solving the reduced-matrix Riccati equation $A^T P + P A - P B R^{-1} B^T P + Q = 0$ (Ogata, 2010).

3.3.4 Control Scheme

The control scheme employed for the swing-up and stabilization of the Furuta pendulum is illustrated in Figure 7. The *Energy Shaping* block computes the control signal u_{ES} according to Equation (34). The *LQR* block computes the control signal u_{LQR} according to (39). The *Switch* block enables the transition of control from u_{ES} to u_{LQR} when the Furuta pendulum approaches the upright equilibrium point. The switching condition is defined as:

$$(|\theta_1| < \delta_1) \wedge (|\theta_2| < \delta_2) \wedge (|\dot{\theta}_1| < \delta_3) \wedge (|\dot{\theta}_2| < \delta_4) \quad (40)$$

The *CPFL* block converts the control signal u into the torque τ according to Equation (19).

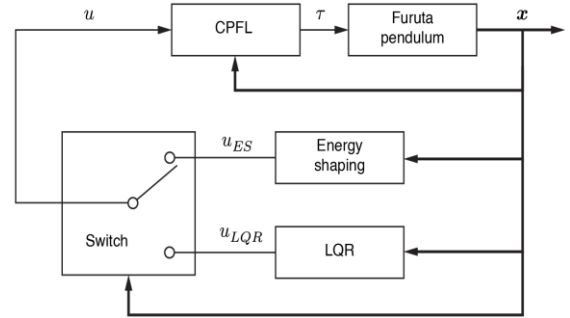


Figure 7 Control scheme for swing-up and stabilization of the Furuta pendulum

4. Results

4.1 Validation of Models

To validate the developed models, a Simulink model was constructed to be used as a reference. For this purpose, the Simscape Multibody libraries were employed. To correctly implement the model, it is essential to define and analyze the reference frames and their transformations within the mechanism (Alvarez & Avalos-Alvarez, 2025). These frames are interconnected through either fixed or variable homogeneous transformations. Figure 8 shows the Simulink model of the Furuta pendulum.

The multibond graph model of the Furuta pendulum (Figure 5) and its analytical dynamic model described by (12) were validated using the previously described Simulink-Simscape model as a reference. The multibond graph model was simulated in the 20-sim software, while the analytical model was implemented in MATLAB. Validation was carried out by comparing the open-loop simulations under a step input. The parameter values employed for the system are reported in Table 1. The simulation conditions were $x_0 = 0$ and $\tau = 0.004$. Figure 9 presents the results for the three models and for both joint variables. The plots are virtually identical. The normalized root mean squared errors (NRMSE) were computed for each pair of models and are reported in Table 2. The maximum NRMSE is 0.2035×10^{-3} .

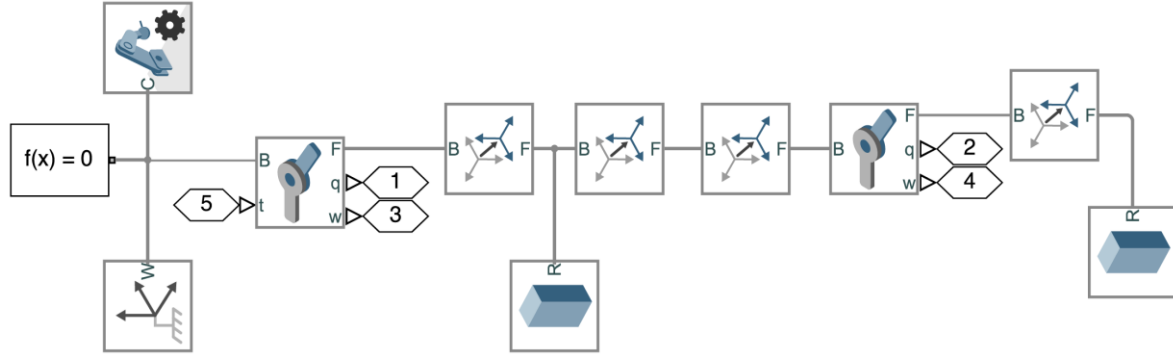


Figure 8 Simulink-Simscape model of the Furuta pendulum

Table 1 Parameters of the Furuta pendulum used in control design and simulation

Parameter	Symbol	Value
Gravitational acceleration	g	9.81 m/s^2
Distance to the center of mass of the rotary arm	a_1	0.125 m
Distance to the center of mass of the pendulum	a_2	0.161 m
Rotary arm length	l_1	0.216 m
Rotary arm mass	m_1	0.257 kg
Pendulum mass	m_2	0.127 kg
Principal moment of inertia of the rotary arm about the y -axis	J_{y1}	$1.01 \times 10^{-3} \text{ kgm}^2$
Principal moment of inertia of the pendulum about the x -axis	J_{x2}	$1.23 \times 10^{-3} \text{ kgm}^2$
Principal moment of inertia of the pendulum about the y -axis	J_{y2}	$8.21 \times 10^{-6} \text{ kgm}^2$
Principal moment of inertia of the pendulum about the z -axis	J_{z2}	$1.23 \times 10^{-3} \text{ kgm}^2$
Viscous friction coefficient at the first joint	b_1	$3 \times 10^{-3} \text{ Nm/(rad/s)}$
Viscous friction coefficient at the second joint	b_2	$3 \times 10^{-3} \text{ Nm/(rad/s)}$

Table 2 NRMSE to compare developed models, for the two joint variables

Pair of Models	NRMSE (10^{-3})	
	θ_1	θ_2
Bond Graph / Simscape	0.1417	0.1844
Analytic / Simscape	0.1441	0.2035
Bond Graph / Analytic	0.0229	0.1072

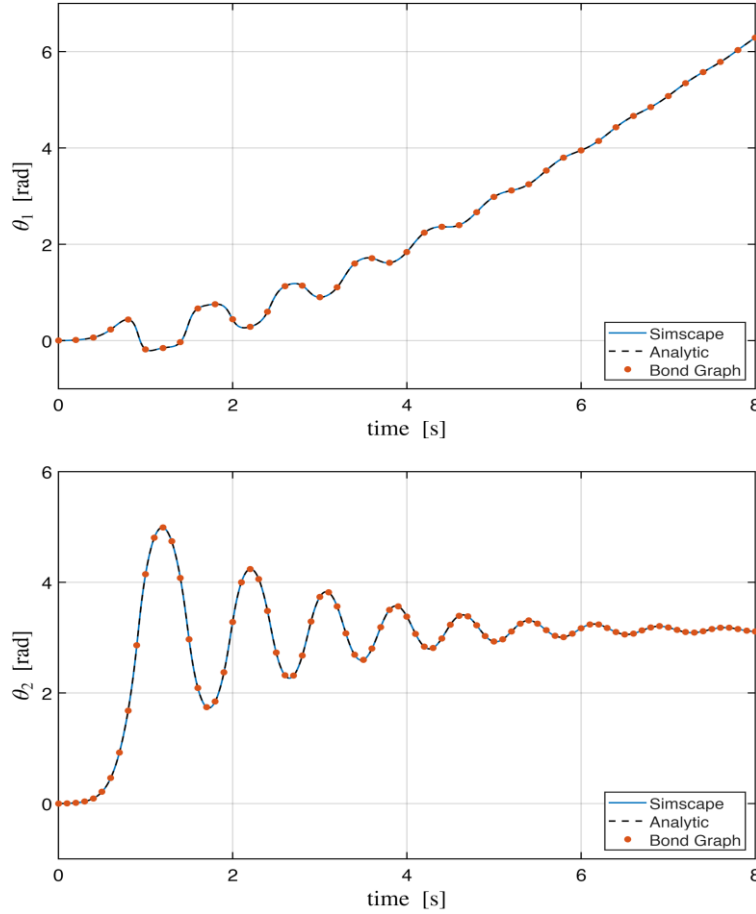


Figure 9 Validation of the developed models in comparison with a Simulink-Simscape model. The simulation conditions were $x_0 = 0$ and $\tau = 0.004$

4.2 Case Studies

This study aimed to evaluate the extent to which controllers designed using simplified models preserve adequate performance when applied to the full-dynamics model. The first model neglects the pendulum's secondary inertias (J_{2x} and J_{2y}), whereas the second model includes them. To quantify the impact of these secondary inertias, the full-dynamics model was evaluated under three conditions: using the nominal inertia values reported in Table 1, and with the secondary inertias scaled by factors of two and four. These variants are denoted as *Full Inertia 1x*, *Full Inertia 2x*, and *Full Inertia 4x*, respectively. The model that neglects the secondary inertias is referred to as the *Simplified* model.

Two case studies are presented. In the first, the models are compared under open-loop conditions in

response to a constant input. In the second, the models are evaluated under closed-loop operation using the control scheme described in Section 3.3.4.

4.2.1 Open-loop Comparison

Through simulations, the models were compared in open loop under a constant input of $\tau = 0.4$ with initial conditions $x_0 = 0$. The results for θ_1 and θ_2 are presented in Figure 10. For θ_1 , the responses exhibit a similar shape; however, as the inertia increases, the response becomes less pronounced. For θ_2 , the responses are also similar, with the first peak occurring at nearly the same time across all four models. Nevertheless, higher inertia results in faster oscillations. The NRMSE values, computed by comparing the full-inertia models against the simplified model, are reported in Table 3. These values increase with inertia.

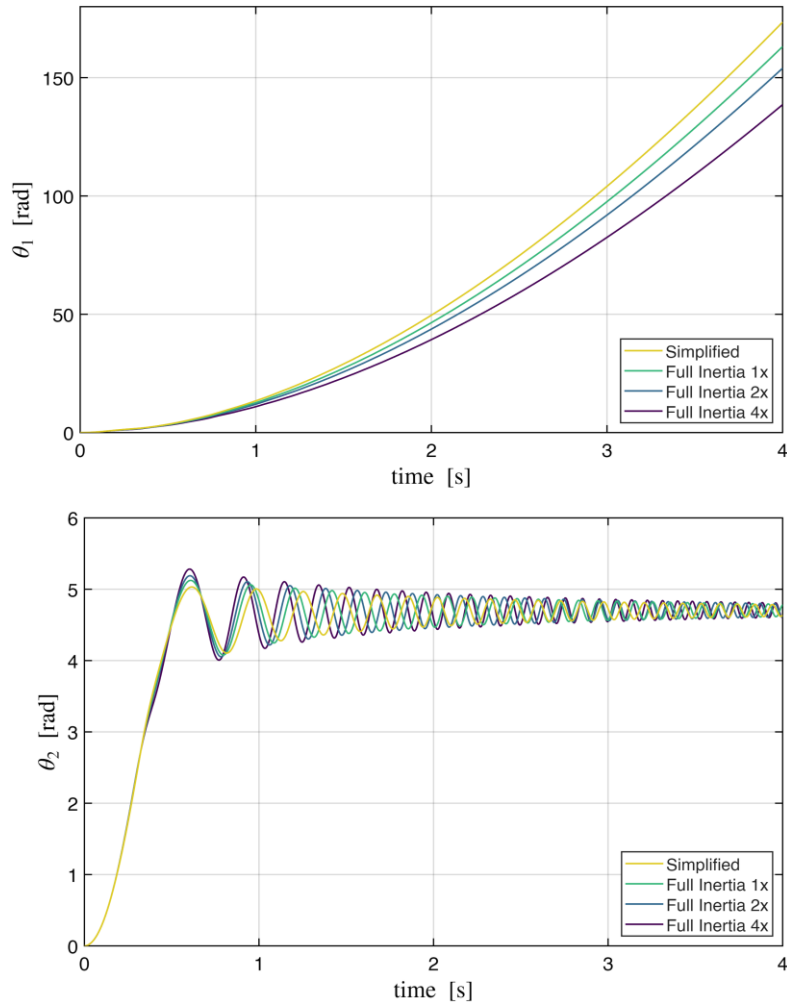


Figure 10 Open-loop step response comparison between the simplified model and full-inertia models under three configurations: nominal J_{x2} and J_{y2} values (as reported in Table 1), doubled values, and quadrupled values.

Table 3 NRMSE values comparing full-inertia models against the simplified model for the two joint variables

Pair of Models	NRMSE (10^{-3})	
	θ_1	θ_2
Full Inertia 1x / Simplified	28.7	37.4
Full Inertia 2x / Simplified	53.9	43.3
Full Inertia 4x / Simplified	96.0	48.9

Table 4 Controller performance for the simplified model and the three full-inertia models

Model	Energy Error ($\text{J}\sqrt{\text{s}}$)	Control Effort ($\text{N}^2\text{m}^2\text{s}$)	Switching Time (s)	Settling time (s)	# of oscillations
Simplified	0.111	0.453	3.92	7.48	2
Full Inertia 1x	0.111	0.461	3.60	7.42	2
Full Inertia 2x	0.111	0.475	3.36	7.44	2
Full Inertia 4x	0.116	0.655	4.86	8.91	3

4.2.2 Closed-loop Comparison

The models were compared in closed loops using the control scheme described in Section 3.3.4. The energy-based control parameters were $k_p = k_D = k_V = 1$, and $\epsilon = 0.001$. The LQR control parameters were $Q = \text{diag}(1, 10, 1, 10)$ and $R = 0.001$, resulting in $K = [-31.6 \ 866 \ -46.3 \ 159]$. The switching condition was defined by $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0.5$. The pendulum was initialized at rest near the downward equilibrium position, specifically with $\theta_1 = 0$ and $\theta_2 = 0.99\pi$. The results for θ_1 , θ_2 , and τ are presented in Figure 11. In all cases, the control system was able to achieve swing-up and stabilize the pendulum at the unstable upright equilibrium position in less than 10 seconds.

The performance of the control system was evaluated using five indicators. Table 4 presents the L^2 norm of the energy error $\tilde{E}$ over the interval $t \in [0, 10]$, the control effort of the input τ over the same interval, the switching time to the LQR controller, the settling time within $\pm 0.02\pi$ for both angles, and the number of pendulum oscillations until stabilization.

Overall, the responses of θ_1 , θ_2 , and τ exhibit notable differences, although they follow similar patterns. The *Simplified*, *Full Inertia 1x*, and *Full Inertia 2x* models required only two oscillations to achieve stabilization, whereas the *Full Inertia 4x* model required three. It is noteworthy that, as inertia increases, the oscillation frequency of the pendulum also increases. In fact, excluding the *Full Inertia 4x* model, the switching time decreases as inertia increases; however, since the *Full Inertia 4x* model undergoes an additional oscillation, this trend no longer holds. The *Simplified*, *Full Inertia 1x*, and *Full Inertia 2x* models exhibited similar values in terms of energy error, control effort, and settling time, while the *Full Inertia 4x* variant showed more significant deviations.

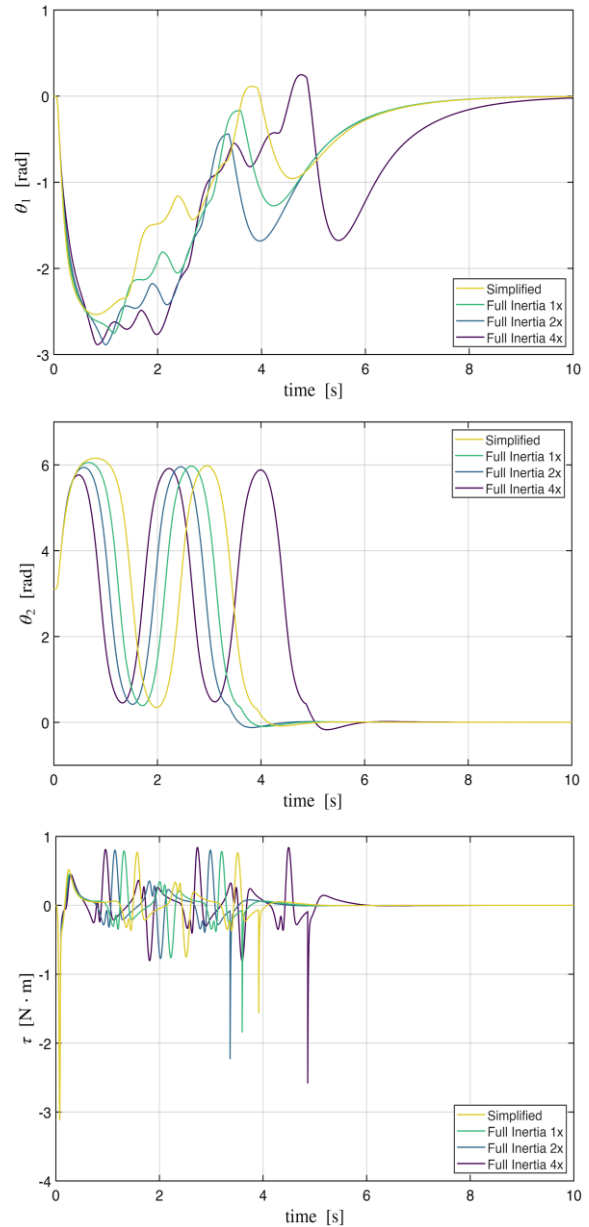


Figure 11 Closed-loop response comparison for swing-up and stabilization control between the simplified model and full-inertia models under three configurations: nominal J_{x2} and J_{y2} values (as reported in the Table 1), doubled values, and quadrupled values

5. Discussion

The main objective of this study was to investigate whether controllers designed based on simplified models of the Furuta pendulum remain effective when applied to full-dynamics models. Specifically, the effect of the pendulum's secondary inertias (J_{2x} and J_{2y}) on control performance was quantified. The results provide valuable insights into the limitations of theoretical abstraction when designing control systems for highly nonlinear underactuated mechanisms.

Open-loop simulations demonstrate that neglecting the secondary inertias alters the natural dynamics of the system. As shown in Figure 10 and Table 3, the NRMSE increases as the unmodeled inertia is artificially scaled from 1x to 4x. This behavior can be attributed to the combined effect of the pendulum and arm inertias, which jointly resist the applied torque; consequently, for a given torque input, an increase in the total inertia leads to a reduction in angular acceleration.

On the other hand, the closed-loop results (Figure 11 and Table 4) present a highly encouraging scenario for control engineers. The control scheme designed using the simplified model exhibited remarkable robustness when applied to the full-dynamics models. For the nominal exact model (*Full Inertia 1x*) and even the doubled inertia model (*Full Inertia 2x*), the degradation in control performance was negligible. The controller successfully completed the swing-up and stabilization maneuvers with the same number of oscillations, virtually identical settling times (~ 7.4 seconds), and minimal increases in control effort compared to the simplified model. This resilience indicates that the unmodeled dynamics introduced by J_{2x} and J_{2y} can be effectively compensated for by the control system. In typical laboratory implementations of the Furuta pendulum, the pendulum is often approximated as a slender rod, for which $J_{2x} = J_{2z}$ and J_{2y} is negligible, as represented by the *Full Inertia 1x* model. Therefore, the common practice of employing the simplified model for control design is, to a certain extent, justified. Whether these results can be generalized to other control strategies remains an open question for future research.

Nevertheless, when the secondary inertias are significantly large (*Full Inertia 4x*), the control performance decreases appreciably. The closed-loop system required an additional oscillation to achieve the upright position, the settling time increased by nearly 20% (to 8.91 s), and the control effort spiked

significantly. In such physical configurations, controllers designed based on the simplified model risk failing to stabilize the system in a real-world scenario.

Beyond the dynamic analysis, this study highlights the practical advantages of multibond graphs over traditional modeling frameworks. While Euler–Lagrange becomes algebraically cumbersome with spatial multi-axis dynamics or dissipative effects, bond graphs provide an explicit, topology-based representation of power exchange. Crucially, this inherent physical consistency ensured the systematic derivation of the exact dynamic mathematical model, enabling the identification of discrepancies in a previously published work. Moreover, this transparent energy mapping directly facilitated the energy shaping control design, as kinetic and potential energies were straightforwardly extracted. The approach also offers exceptional modularity; multibond graphs seamlessly expand to multi-link systems or integrate multi-domain elements like Direct Current (DC) motors. Coupled with software like 20-sim, this framework is a highly scalable tool for mechatronic design.

While the multibond graph model provides a highly accurate and scalable virtual testbed, it is important to acknowledge its current limitations. By treating the components as perfectly rigid and the joints as ideal, the model cannot capture structural vibrations, compliances, or mechanical backlash. Furthermore, the assumption of instantaneous torque delivery and purely viscous friction omits practical nonlinearities such as motor saturation, communication delays, or stiction. These simplifications could challenge the controller's robustness and limit the direct translation of simulated control efforts to real-world hardware. Future work will focus on integrating these constraints into the bond graph model and performing experimental validation on a physical prototype.

6. Conclusion

This study analyzed the impact of modeling simplifications on the dynamics and control performance of the Furuta pendulum, focusing on the role of secondary inertias, which are often neglected. A complete multibond graph model was developed, providing an energetically consistent, modular, and scalable framework that serves as a reliable reference for validation. This structured representation, along with the subsequent mathematical formulation,

enabled the identification of errors in a previously published dynamic model.

The results show that neglecting secondary inertias appreciably affects the open-loop dynamics, with discrepancies increasing as these inertial terms grow. However, in closed-loop operation, the control scheme based on energy shaping, collocated partial feedback linearization (CPFL), and linear quadratic regulation (LQR) designed using the simplified model demonstrates strong robustness. For nominal and moderately increased inertia conditions, the performance degradation is minimal, with similar settling times, control effort, and oscillatory behavior.

Nevertheless, when the neglected inertias become sufficiently large, the control performance deteriorates, leading to longer settling times, higher control effort, and additional oscillations. This indicates that the validity of simplified models is limited to moderate deviations from the true dynamics.

Overall, simplified models are adequate for control design in typical scenarios, while full-dynamics models remain essential for validation and for assessing robustness limits. Future work will focus on incorporating non-ideal effects such as actuator dynamics, nonlinear friction, and structural flexibility into the model, as well as performing experimental validation on a physical platform.

7. Acknowledgements

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8. Abbreviations

Abbreviation	Full Term
CPFL	Collocated Partial Feedback Linearization
DC	Direct Current
EJS	Eulerian Junction Structure
LQR	Linear Quadratic Regulator
NRMSE	Normalized Root Mean Squared Error
PD	Proportional Derivative
PID	Proportional Integral Derivative

9. CRediT Statement

Paul Alvarez: Conceptualization, Methodology, Software, Validation, Formal Analysis, Investigation, Writing – Original Draft, Writing – Review & Editing, Supervision.

Alexander Rodríguez Gamboa: Methodology, Software, Investigation, Resources, Data Curation,

Writing – Review & Editing, Visualization, Project Administration.

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